
Contents

Preface	ix
Mathematicians, Physicists and Astronomers	
Cited in this Book	xiii
Important Notations	xv
Part 1. Introduction to the Calculus of Variations	1
Chapter 1. Elementary Methods to the Calculus of Variations	3
1.1. First free extremum problems	3
1.2. First constrained extremum problem – Lagrange multipliers	7
1.2.1. Example of Lagrange multiplier	7
1.2.2. Application to the constrained extremum problem	8
1.3. The fundamental lemma of the calculus of variations	10
1.4. Extremum of a free functional	11
1.5. Extremum for a constrained functional	13
1.5.1. First type: integral constraint	13
1.5.2. Second type: distributed constraint	16
1.6. More general problem of the calculus of variations	19
1.6.1. First extension of the earlier results	19
1.6.2. Important example	19
1.6.3. First application: the Hamilton principle	21
1.6.4. Second application: geodesics of surfaces	23
Chapter 2. Variation of Curvilinear Integral	29
2.1. Geometrization of variational problems	29
2.2. First form of curvilinear integral	31

2.3. Second form of curvilinear integrals	35
2.4. Generalization and variation of derivative	37
2.5. First application: studying the optical path of light	41
2.5.1. Fermat's principle	41
2.5.2. Descartes' laws	44
2.6. Second application: the problem of isoperimeters	46
Chapter 3. The Noether Theorem	51
3.1. Additional results on differential equations	51
3.2. One-parameter groups and Lie groups	53
3.3. Invariant integral under a Lie group	56
3.4. Further examination of Fermat's principle	58
Part 2. Applications to Analytical Mechanics	61
Chapter 4. The Methods of Analytical Mechanics	63
4.1. D'Alembert's principle	63
4.1.1. Concept of virtual displacement	63
4.1.2. Concept of constraints	64
4.1.3. The Lagrange formulae	65
4.2. Back to analytical mechanics	68
4.3. The vibrating strings	69
4.3.1. First study of the solutions of equation [4.11]	71
4.3.2. Second study of the solutions of equation [4.11]	72
4.4. Homogeneous Lagrangian. Expression in space time	75
4.5. The Hamilton equations	78
4.5.1. First method using Lagrange equations	78
4.5.2. Second method using the Hamilton principle	80
4.6. First integral by using the Noether theorem	83
4.6.1. Secondary parameters	83
4.6.2. Back to the Noether theorem	84
4.7. Re-injection of a partial result	88
4.8. The Maupertuis principle	90
4.8.1. First application: case of a material point	91
4.8.2. Second application: introduction to the Riemannian geometry	92
Chapter 5. Jacobi's Integration Method	95
5.1. Canonical transformations	95
5.2. The Jacobi method	98
5.2.1. One position parameter is secondary	100
5.2.2. Time is a secondary parameter	101
5.3. The material point in various systems of representation	101

5.3.1. Case of Cartesian coordinates	101
5.3.2. Case of cylindrical representation	102
5.3.3. Case of spherical representation	104
5.4. Case of the Liouville integrability	105
5.5. A specific change of canonical variables	107
5.6. Multi-periodic systems. Action variables	109
Chapter 6. Spaces of Mechanics – Poisson Brackets	113
6.1. Spaces in analytical mechanics	113
6.2. Dynamical variables – Poisson brackets	116
6.2.1. Evolution equation of a dynamical variable	116
6.2.2. First integral	118
6.3. Poisson bracket of two dynamical variables	118
6.3.1. Properties of Poisson brackets	119
6.3.2. Application to the Noether theorem	121
6.4. Canonical transformations	123
6.4.1. Calculus of the Poisson matrix	125
6.5. Remark on the symplectic scalar product	131
Part 3. Properties of Mechanical Systems	133
Chapter 7. Properties of Phase Space	135
7.1. Flow of a dynamical system	135
7.2. The Liouville theorem	138
7.2.1. Preliminary	138
7.2.2. Application to mechanical systems	141
7.3. The Poincaré recurrence theorem	144
7.3.1. The recurrence theorem	144
7.3.2. Case of mechanics	145
Chapter 8. Oscillations and Small Motions of Mechanical Systems	149
8.1. Preliminary remarks	149
8.2. The Weierstrass discussion	151
8.2.1. Introduction	151
8.2.2. Discussion of fundamental equation [8.3]	152
8.2.3. Graphical interpretation	155
8.2.4. Study of equilibrium positions	157
8.2.5. Small motions near an equilibrium position	159
8.3. Equilibrium position of an autonomous differential equation	162
8.4. Stability of equilibrium positions of an autonomous differential equation	164
8.5. A necessary condition of stability	164

8.6. Linearization of a differential equation	169
8.6.1. Preliminary	169
8.6.2. Application to Lagrangian dynamical systems	171
8.6.3. Small oscillations of a Lagrangian dynamical system	173
8.7. Behavior of eigenfrequencies	177
8.7.1. Preliminaries	177
8.7.2. Behavior of eigenfrequencies with the system rigidity	177
8.7.3. The frequencies' behavior when parameters are linked by constraints	179
8.8. Perturbed equation associated with linear differential equation	181
Chapter 9. The Stability of Periodic Systems	187
9.1. Position of the problem	187
9.2. Flow of a periodic differential equation	188
9.3. Study of the planar case	190
9.3.1. Generalities	190
9.3.2. Case where $\det \mathcal{G} = 1$	191
9.4. Strong stability in periodic Hamiltonian systems	192
9.5. Study of the Mathieu equation. Parametric resonance	193
9.6. A completely integrable case of the Hill equation	195
Part 4. Problems and Exercises	201
Chapter 10. Problems and Exercises	203
Chapter 11. Solutions to Problems and Exercises	233
References	301
Index	303