Contents

Preface xi

Errata for Volume 1 xvii

List of Notation xix

Chapter 1. Field Extensions and Differential Field Extensions 1

1.1. Galois theory 1
1.1.1. Introduction 1
1.1.2. Overview and terminology 3
1.1.3. Decomposition fields 9
1.1.4. Algebraically closed fields 13
1.1.5. Separability 15
1.1.6. Normality 24
1.1.7. Galois extensions and Galois groups 26
1.1.8. Fundamental theorem of Galois theory 28
1.1.9. Binomial equation 31
1.1.10. Solving algebraic equations by radicals 33
1.1.11. General algebraic equations 36
1.1.12. Topology of the Galois group 36
1.2. Transcendental extensions 37
1.2.1. Pure extensions 38
1.2.2. Transcendence bases 38
1.3. Differential Galois theory 40
1.3.1. Introduction 40

1.3.2. Picard-Vessiot extensions 42
1.3.3. Algebraic groups 44
1.3.4. Fundamental theorem of Picard-Vessiot theory 46
1.3.5. Simple adjunctions 48
1.3.6. Liouville extensions and elementary functions 50
1.4. Differentially transcendental extensions 52
1.4.1. Differentially algebraic dependence 52
1.4.2. Differential transcendence degree 53

Chapter 2. General Topology 55

2.1. Introduction to general topology 55
2.1.1. Metric spaces 55
2.1.2. Concept of topology 57
2.1.3. General topology vs. topology in metric spaces 57
2.2. Filters and nets 60
2.2.1. Filters and ultrafilters 60
2.2.2. Nets and ultranets 62
2.3. Topological structures 65
2.3.1. Elementary concepts 65
2.3.2. Convergence 67
2.3.3. Continuity 70
2.3.4. Category of topological spaces 71
2.3.5. Products and projective limits of topological spaces 73
2.3.6. Coproducts and inductive limits of topological spaces . . . 74
2.3.7. Compactness 75
2.3.8. Connectedness 78
2.3.9. Locally compact spaces 79
2.3.10. Paracompact spaces 82
2.3.11. Normal spaces 83
2.3.12. Continuous partitions of unity 83
2.3.13. Germs of mappings 84
2.4. Uniform structures 84
2.4.1. Entourages and uniformizable topologies 84
2.4.2. Uniform continuity 87
2.4.3. Families of pseudometrics and Lipschitz structures 88
2.4.4. Completeness 91
2.4.5. Uniformity of compact spaces 95
2.5. Bornologies 98

2.5.1. Concept of bornology 98
2.5.2. Compactness in $\mathbb{K}^n$ 100
2.5.3. Boundedness of filters and nets 100
2.6. Baire spaces, Polish spaces, Suslin spaces, and Lindelöf spaces . 101
2.6.1. Baire spaces and Baire's category theorem 101
2.6.2. Polish spaces and Suslin spaces 102
2.6.3. Lindelöf spaces 103
2.7. Uniform function spaces 103
2.7.1. $\mathfrak{S}$-convergence 103
2.7.2. Sets of continuous mappings 105
2.7.3. Equicontinuity 106
2.7.4. Equiboundedness 109
2.8. Topological algebra 109
2.8.1. Topological groups 109
2.8.2. Topological abelian groups 115
2.8.3. Topological rings and modules, etc. 116

Chapter 3. Topological Vector Spaces 117

3.1. Introduction 117
3.2. General topological vector spaces 118
3.2.1. Topologies of topological vector spaces 118
3.2.2. Category of topological vector spaces 120
3.2.3. Riesz and Banach theorems 124
3.3. Locally convex spaces 126
3.3.1. Convex sets and functions 126
3.3.2. Semi-norms 127
3.3.3. Locally convex topologies 128
3.3.4. Hahn-Banach theorem 129
3.3.5. Category of locally complex spaces 132
3.3.6. Bounded sets 133
3.3.7. Topological direct sums and inductive limits 134
3.3.8. $\mathfrak{S}$-topology on $\mathcal{L}(E;F)$ 136
3.4. Important types of locally convex spaces 137
3.4.1. Banach spaces 137
3.4.2. Fréchet spaces 141
3.4.3. Barreled spaces 142
3.4.4. Bornological spaces and ultrabornological spaces 145
3.4.5. Montel spaces 147

3.4.6. Schwartz spaces 148
3.5. Weak topologies 151
3.5.1. Dual systems 151
3.5.2. Bipolars 152
3.5.3. Transpose of a linear mapping 153
3.6. Dual of a locally convex space 156
3.6.1. Notion of a dual space 156
3.6.2. Weak* topology on E' 156
3.6.3. Topologies compatible with duality 158
3.6.4. Other topologies on E' 159
3.6.5. Riesz spaces 162
3.7. Bidual and reflexivity 163
3.7.1. Embedding of E within its bidual 163
3.7.2. Semi-reflexive spaces 163
3.7.3. Reflexive spaces 164
3.7.4. Duality in Banach spaces 167
3.7.5. Strong dual of a Montel space 168
3.8. Additional notes about $(\mathcal{F})$ and $(\mathcal{FS})$-spaces and their duals . . 169
3.8.1. Strict morphisms of Fréchet spaces 169
3.8.2. Strict morphisms of $(\mathcal{FS})$ or Silva spaces 170
3.9. Continuous multilinear mappings 174
3.9.1. Continuous bilinear mappings 174
3.9.2. Hypocontinuous bilinear mappings 175
3.9.3. Bounded multilinear mappings 176
3.10. Hilbert spaces 178
3.10.1. Pre-Hilbert spaces 178
3.10.2. Hilbert spaces 180
3.10.3. Operators in Hilbert spaces 188
3.11. Nuclear spaces 191
3.11.1. Tensor product topologies 192
3.11.2. Nuclear mappings 193
3.11.3. Nuclear spaces 194

Chapter 4. Measure and Integration, Function Spaces 197

4.1. Measure and integration 197
4.1.1. “Abstract” measures 198
4.1.2. Integration and L^p-spaces 201
4.1.3. Tensor product of measures 207

4.1.4. Spaces of continuous functions 209
4.1.5. Radon measures 210
4.1.6. Integral with respect to a Radon measure 214
4.1.7. Absolutely continuous functions 217
4.1.8. Continuous sums of Hilbert spaces 223
4.2. Functions in a single complex variable 223
4.2.1. Holomorphic functions in a single complex variable 223
4.2.2. Homology. Cauchy's theorem 225
4.2.3. Cauchy's integral formula 227
4.2.4. Analytic functions in a single complex variable 228
4.2.5. Residue theorem 229
4.2.6. Meromorphic functions 230
4.2.7. Argument principle 233
4.3. Function spaces 234
4.3.1. Spaces of infinitely differentiable functions 234
4.3.2. Spaces of analytic functions 238
4.4. Generalized function spaces 242
4.4.1. Distributions 242
4.4.2. Hyperfunctions on the real line 247

Chapter 5. Sheaves 257

5.1. Introduction 257
5.2. General results about sheaves 258
5.2.1. Presheaves 258
5.2.2. Sheaves 261
5.2.3. Fibered spaces and étalé spaces 266
5.2.4. Examples 270
5.2.5. Sheaves of germs of homomorphisms 271
5.2.6. Projective limits of sheaves 272
5.2.7. Inductive limits of sheaves 273
5.2.8. Sheaves taking values in an abelian category 275
5.3. Sheaves of Modules 275
5.3.1. Modules over a ringed space 275
5.3.2. Coherent sheaves 278
5.3.3. Examples of coherent sheaves 281
5.4. Cohomology of sheaves 285
5.4.1. Acyclic, fine, soft, flabby sheaves 285
5.4.2. Cohomology of coherent sheaves 287

5.4.3. Relative cohomology 289
5.4.4. Meromorphic functions in multiple variables 292
5.4.5. Sheaf of hyperfunctions 297
5.4.6. Application to systems of linear differential equations . . . 299

Bibliography 307

Cited Authors 313

Index 321