Chapter 3

Homogenization by Multiple Scale Asymptotic Expansions

3.1. Introduction

Following discussion of the multiple scale method and its formalism, in this chapter we will explain in detail how it can be implemented. We will begin by using basic experiments to show how the concepts presented in the previous chapters apply in reality and how they match up to physical intuition. We will then show how the homogenization process is carried out for a one-dimensional example with an analytical solution. Finally, the last section focuses on the translation of physical problems into the framework of the multiple scale method.

3.2. Separation of scales: intuitive approach and experimental visualization

The concept of multiple scales, and its use in homogenization methods, may appear an abstract one that could be taken as a mathematical trick. It is no such thing, because in fact this idea represents an actual physical reality. Here we will try to help the reader grasp this by using an intuitive approach illustrated with simple experimental examples.

3.2.1. *Intuitive approach to the separation of scales*

We have already seen that homogenization involves the search for a given phenomenon in a given heterogenous material, for an equivalent – or “homogenized” – global description, which does not make any explicit reference to local fluctuations.

This statement incorporates, as a subtext, the concept of separation of scales, since a global description has no meaning if the phenomenon of interest only varies on a local scale. As indicated in the preceding chapters, it is this crucial concept of separation of length scales which makes it possible to look for a homogenized description. The concept can be described in terms of two requirements:

– the first involves the medium, which must be such that we can define a characteristic length l, which is only possible if the material has a representative elementary volume (without a REV, there is no characteristic length!);

– the second involves the phenomenon: a quantity associated with it must exhibit a characteristic length L, which is large compared to l.

The graphics in Figure 3.1 (top) give a visual depiction of the separation of scales required for homogenizability: as long as the phenomenon of interest (the runner) has a scale of motion (his stride) which is large compared to the REV of the material (a sand or pebble beach), a global description (of the speed and trajectory of the runner) which ignores local fluctuations (the exact positions of the grains of sand or pebbles) is possible. It can be clearly seen in this example that for the phenomena involved a homogenized description is more efficient – and also more realistic – than a description which incorporates every last detail of reality without removing all but the essential parts of the picture.

Outside this framework, in other words without a separation of scales, the search for a macroscopic description is doomed to failure (Figure 3.1 below): on a route consisting of meter-sized rocks, neither the trajectory nor the speed of the runner can be known independently of the distribution of the blocks. This would also be the case for an insect on the pebble beach (despite the fact that it is homogenizable for the runner). This illustrates the fact that homogenizability is a property not intrinsic to the material or the phenomenon, but which depends on the material/phenomenon pair.

The role of the periodic or random nature of the microstructure (when the constituents follow the same connectivity conditions) was discussed in Chapter 2. The images in Figure 3.2 (top) illustrate the main conclusions: when the separation of scales is obvious, whatever the organization on the local scale (pebbles arranged periodically or laid out randomly), the mechanisms (the determination of the runner's trajectory) are the same, and as a result the macroscopic behavior of the material (what the runner experiences) will be qualitatively the same. Here we justify the use of the method of periodic media for treating real aperiodic materials when there is a separation of scales. Conversely, the closer the macroscopic scale gets to the microscopic scale, the more sensitive it becomes to local fluctuations, and consequently the organization of the microstructure. At the limit of the homogenizable domain, Figure 3.2 (bottom – the runner striding across separate blocks), the

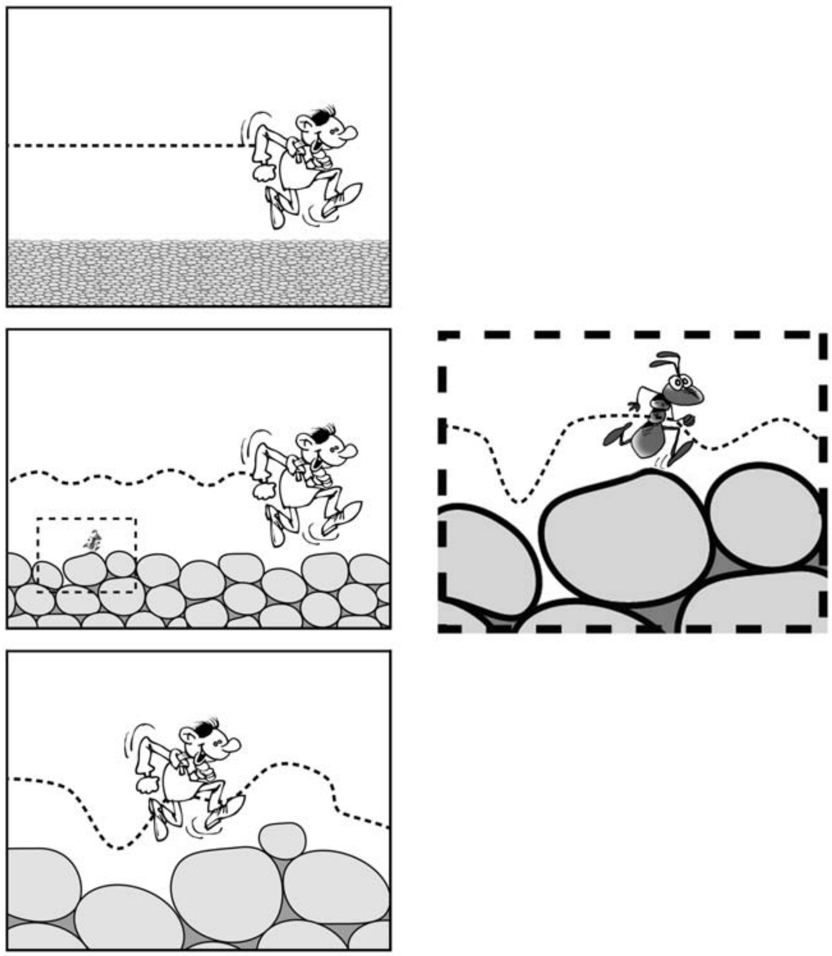


Figure 3.1. *The separation of scales is the sine qua non condition for a global description. Here it is only the case in the top picture: the property of homogenizability only has a meaning for the combination of the material and the phenomenon together (illustrations by Jacques Sardat)*

phenomena in periodic media (where the runner can jump from block to block) and random media (where the runner falls between blocks that are too far apart) diverge: without the separation of scales, homogenization loses its meaning, and the type of organization within the microstructure becomes critical.

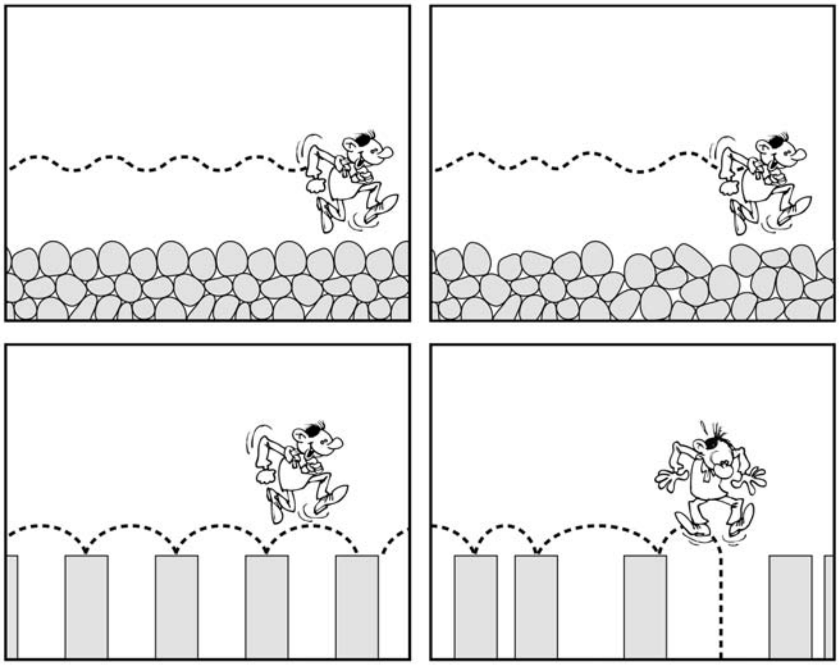


Figure 3.2. *The role of the microstructure layout is more significant when there is not a good separation of scales (illustrations by Jacques Sardat)*

3.2.2. *Experimental visualization of fields with two length scales*

Here we will investigate, with the help of basic experiments on two periodic two-dimensional media: the manifestation of local and global scale variations, and the (quasi-)periodicity or local periodicity of the fields.

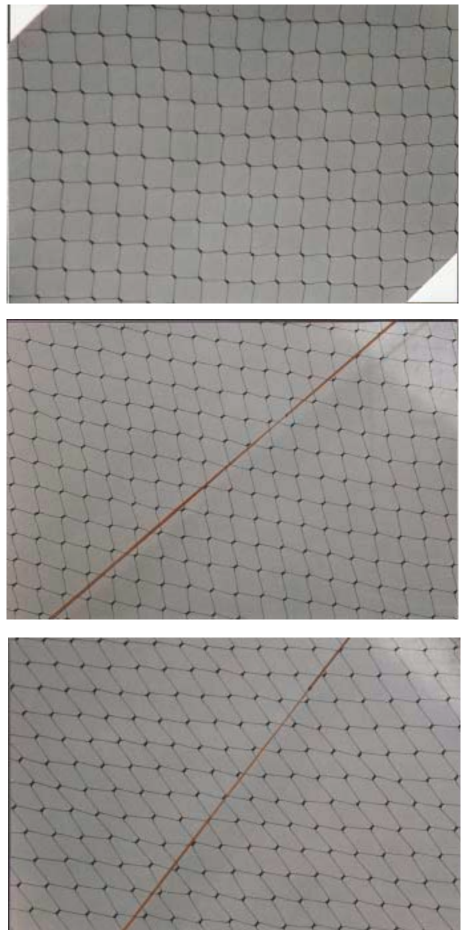
3.2.2.1. *Investigation of a flexible net*

The photos in Figure 3.3 show a net with a diamond mesh (period Ω) fixed at its edges to a square framework consisting of four rigid, articulated rods. If we apply a distortion to the frame, we impose a homogenous distortion to the net:

– photo (a) in Figure 3.3 shows the starting position, where the net is undistorted;

– photos (b) and (c) in Figure 3.3 show the geometries obtained when a moderate, and then considerable, distortion is applied to the supporting framework. It is clear that the mesh is distorted but that the structure remains periodic.

So, for homogenous deformations, the property of periodicity of the initial medium is preserved by the perturbations, even for large deformations.



(a) (b) (c)

Figure 3.3. *Visualization of a periodic net (a) of the periodicity under (b) moderate and (c) large deformations*

What happens for loadings which lead to inhomogenous distortions?

– In photo (a) in Figure 3.4 the net is dragged in the plane by a rigid rake which applies a tension across several units of the mesh. The deformation produced in the net is not homogenous. Nevertheless, a local (quasi-)periodicity (i.e. Ω-periodicity relative to the microscopic variable) is visible. What we mean is that all the meshes adjacent to mesh A have an almost identical geometry. The same is true of mesh B. However, meshes A and B, which are fairly far apart from each other, have a very different geometry. We also point out that the geometries of the deformed meshes are the same as we have already seen under homogenous distortion. This situation shows the two-length-scale variations, where each cell is deformed, but the amplitude of this deformation varies gradually across distances corresponding to many mesh cells.

– If instead of being spread out the tension is applied at one point (photograph (b) in Figure 3.4), a new effect appears which is characterized by a violation of the local periodicity on either side of the line of the pull. In these regions where there is a high gradient of deformation, there is no longer a separation of scales because the phenomenon is concentrated on the local scale (which leads to the loss in periodicity): perpendicular to the direction of the pull the problem is not homogenizable.

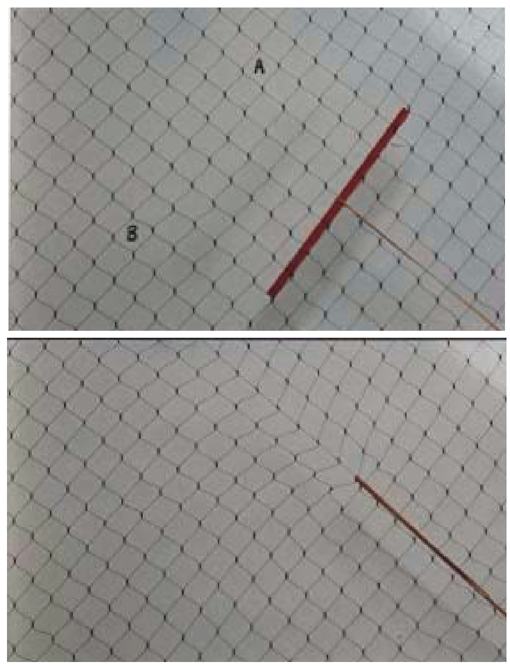


Figure 3.4. *Inhomogenous load: the quasi-periodicity relies on a separation of length scales: (a) a load which respects the separation of scales: the local quasi-periodicity is modulated by large-scale variations; (b) localized loading: the periodicity is lost along the line of the pull*

This net makes it possible to directly observe the deformed geometry of the lattice. However, in many homogenization problems the period is considered to be fixed, as is effectively the case of flow in rigid porous media, cases of heat transfer or diffusive solute transport, etc. or where it is an approximation which can be justified by the low level of deformation such as when considering elastic composites, poroelastic behavior, etc. In this case the homogenizability conditions apply to the fields which develop within this periodic geometry. We will consider such a case in the next example.

3.2.2.2. *Photoelastic investigation of a perforated plate*

Consider a plexiglass plate drilled with oblong windows distributed in a periodic staggered pattern, which we will subject to small deformations in plane. Through the photoelastic effect we can visualize the deviatoric stresses which develop in the plate under different loads:

– when the plate is loaded uniformly in its plane (photograph (a) in Figure 3.5) it is very obvious that the field is periodic, matching up with the periodicity of the plate;

– if the loading area on the upper edge is reduced (photograph (b) in Figure 3.5) while maintaining the entire contact surface on the lower edge, the local quasi-periodicity and the global fluctuations can both be seen;

– "large-scale" intensity variations are even clearer in photograph (c) in Figure 3.5 where the load is pointlike. It is clear in this case that close to where the force is applied the phenomenon is not homogenizable, but that it becomes so outside a region around the point of loading (which extends for around one period). We will return to this aspect of the problem at the end of this chapter.

We also remark that outside the areas of concentrated load, the local distribution of the deviatoric stresses looks the same, but its global evolution depends on the load.

If on the other hand the same load is applied in a different orientation relative to the plate (photographs (a) and (b) in Figure 3.6) the local and global distributions of the deviatoric stresses are completely changed (but of course the periodicity is still retained). Hence the anisotropy of the distribution of the perforations has a direct impact on the local, and hence global, strains: this illustrates the fact that the macroscopic description is tightly linked to the microscopic structure.

Also, across all the pictures, an edge effect can be seen at the boarder of the plate, which rapidly fades towards the middle of the periodic medium. This rapid decrease can be confirmed in photograph (c) of Figure 3.6 where, under homogenous compression, the periodicity remains obvious when the plate only consists of one-and-a-half periods! The edge effects which result from the loss of periodicity at the boundary can be treated from a theoretical point of view by the introduction of a boundary layer [see for example SAN 87; AUR 87a].

In conclusion, these two examples show how the principles of homogenization have a basis in physical reality. They also show that these principles apply even some way from the ideal separation of scales which the theoretical developments require. Indeed, in situations of inhomogenous loading, the actual scale ratio ε_r is, at very best, in the order of the inverse of the number of periods contained in the smallest dimension of the experiment, so $\varepsilon_r \simeq 0.1$ for the net and $\varepsilon_r \simeq 0.3$ for the plate. This possibility of extending the field of applicability is also one of the main reasons

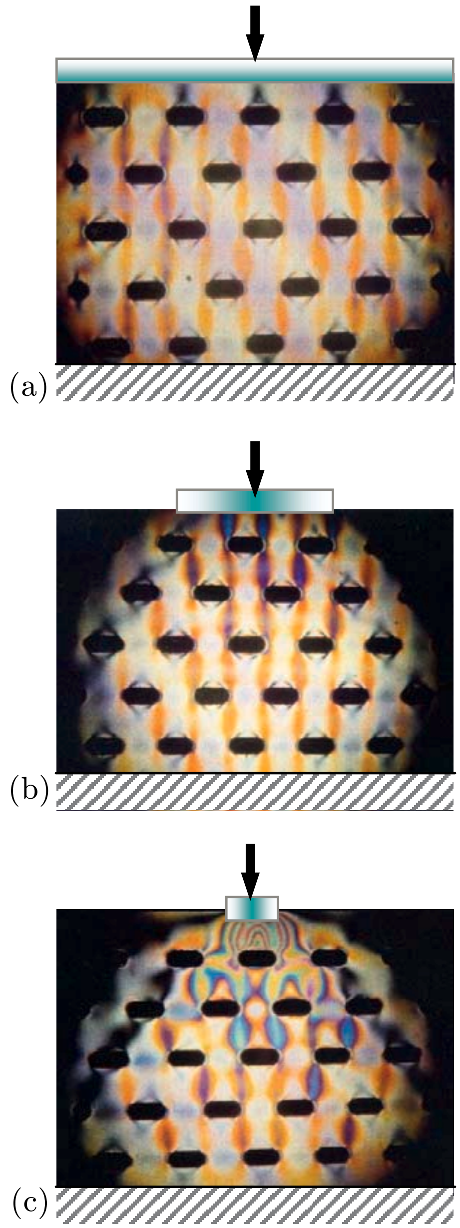


Figure 3.5. *Condition of separation of scales, and quasi-periodicity of the fields in a periodic medium. Perforated plate subject to: (a) homogenous compression exerted by pressure across the width of the plate, (b) inhomogenous compression exerted by a pressure from above across a narrower width, (c) inhomogenous compression under point loading of the upper surface*

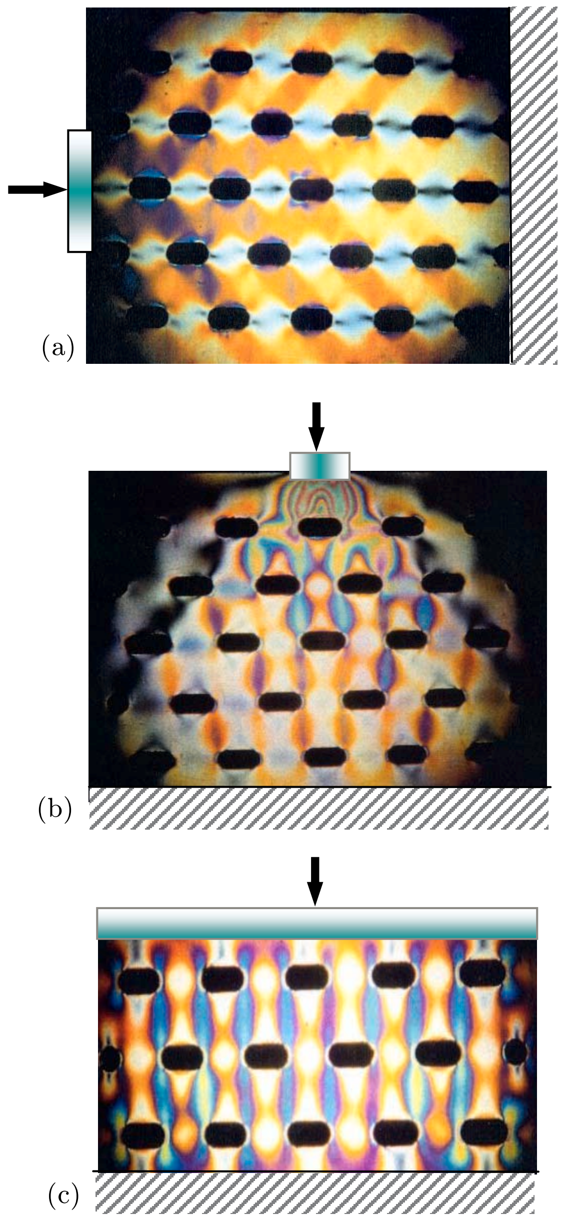


Figure 3.6. *Role of the microstructure in the distribution of local and global forces: (a) inhomogenous compression parallel and (b) perpendicular to the holes, (c) quasi-periodicity and edge effects under homogenous compression for a plate which only consists of one-and-a-half periods*

why the results of homogenization are so good at describing real situations: results established rigorously in the context of ideal assumptions retain their pertinence for real physical situations corresponding to weakened hypotheses. From a theoretical viewpoint, this observation is analogous to proof that the results converge when the scale ratio approaches zero.

To clarify what is meant by this, we will return to these issues, and to the importance of the actual separation of scales in a real-life problem, after we have demonstrated application of the method to a simple example.

3.3. One-dimensional example

Now, and in what follows, we will systematically apply the method of multiple scales, following the methodology laid out in Chapter 2, section 2.4.2. In order to present the various stages of the process, we have selected a one-dimensional example which has an analytical solution. Due to its simplicity, this example cannot include all the problems inherent to homogenization techniques. We will encounter them in subsequent chapters during the study of multi-dimensional problems.

Here we will consider a one-dimensional elastic Galilean medium with an oedometric modulus E and density ρ, subject to a dynamic perturbation. The medium is periodic, with a small period l_c, and we will consider a sample of length $L_c \gg l_c$. The displacement u is governed by the equation of dynamic equilibrium:

$$\mathrm{div}_X(E\ \mathrm{grad}_X u) = \rho \frac{\partial^2 u}{\partial t^2} \tag{3.1}$$

where div_X and grad_X are the divergence and gradient operators with respect to the spatial variable X, which are the same here since the problem is one-dimensional. We recall that $E(X)$ is a positive quantity, as is the density ρ. They are both periodic with period l_c, and may exhibit discontinuities. Figure 3.7 shows an example of the variation of E.

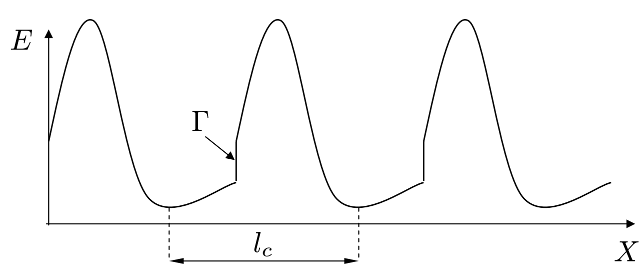


Figure 3.7. *Periodic variation of E*

Across the discontinuities Γ, the stress σ and the displacement u are continuous:

$$[\sigma] = [E \text{ grad}_X u]_\Gamma = 0 \tag{3.2}$$

$$[u]_\Gamma = 0 \tag{3.3}$$

In the above equations, $[\phi]_\Gamma$ indicates the jump in ϕ across the interface Γ. We will first consider the steady state problem, where the second member of (3.1) is zero. We will then treat the dynamic case.

3.3.1. *Elasto-statics*

Equation (3.1) now becomes:

$$\text{div}_X(E \text{ grad}_X u) = 0 \tag{3.4}$$

Equations (3.2, 3.3, 3.4) do not introduce any dimensionless numbers. We will take the microscopic viewpoint. With:

$$X = l_c y^*, \qquad E = E_c E^*, \qquad u = u_c u^*$$

where E_c and u_c are characteristic values, we have:

$$\sigma = E \text{ grad}_X u = \sigma_c \sigma^* \quad \text{with} \quad \sigma_c = E_c u_c / l_c$$

Equation (3.4) becomes:

$$\text{div}_{l_c y^*}(E_c E^* \text{ grad}_{l_c y^*} u_c u^*) = 0$$

Carrying out the same change of variables in (3.2, 3.3), we find after simplification of the terms referring to the same characteristic values:

$$\text{div}_{y^*}(E^* \text{ grad}_{y^*} u^*) = 0, \qquad [E^* \text{ grad}_{y^*} u^*]_{\Gamma^*} = 0, \qquad [u^*]_{\Gamma^*} = 0$$

The unkown u^* must be found in the form of the following expansion:

$$u^*(x^*, y^*) = u^{*(0)}(x^*, y^*) + \varepsilon u^{*(1)}(x^*, y^*) + \cdots, \qquad x^* = \varepsilon y^* \tag{3.5}$$

where $\varepsilon = l_c/L_c$ and $u^{*(i)}$ are periodic with respect to the local variable $y^* = X/l_c$, of period 1. The differential operators are therefore operators with respect to the variable y^*, and in (3.5) $x^* = \varepsilon y^*$. The equivalent macroscopic description will be valid when the perturbation satisfies the condition of separation of scales (we assume that the condition on the separation of geometric scales is met).

3.3.1.1. *Equivalent macroscopic description*

The method involves the introduction of the expansion (3.5) into the dimensionless system and identifying the powers of ε. We note that due to the two spatial variables and the choice of the microscopic viewpoint, the spatial derivative takes the following form:

$$\frac{\partial}{\partial y^*}+\frac{\partial x^*}{\partial y^*}\frac{\partial}{\partial x^*}=\frac{\partial}{\partial y^*}+\varepsilon\frac{\partial}{\partial x^*}$$

The local description becomes:

$$(\frac{\partial}{\partial y^*}+\varepsilon\frac{\partial}{\partial x^*})\left(E^*(y^*)(\frac{\partial}{\partial y^*}+\varepsilon\frac{\partial}{\partial x^*})u^*\right)=0$$

with:

$$[E^*(y^*)(\frac{\partial}{\partial y^*}+\varepsilon\frac{\partial}{\partial x^*})u^*]_{\Gamma^*}=0$$

$$[u^*]_{\Gamma^*}=0$$

across the discontinuities. Substituting the expansion (3.5) into these expressions, we obtain in succession the following results, separating out terms of the same power of ε:

First order in ε^0*:* the system defining $u^{*(0)}$ is the following:

$$\frac{\partial}{\partial y^*}\left(E^*(y^*)\frac{\partial u^{*(0)}}{\partial y^*}\right)=0 \tag{3.6}$$

$$[E^*(y^*)\frac{\partial u^{*(0)}}{\partial y^*}]_{\Gamma^*}=0$$

$$[u^{*(0)}]_{\Gamma^*}=0$$

where $u^{*(0)}$ is 1-periodic in y^*. By successive integration of (3.6) it follows, making use of the conditions at the discontinuities, that:

$$E^*(y^*)\frac{\partial u^{*(0)}}{\partial y^*}=\sigma^{*(0)}(x^*)$$

$$u^{*(0)}(x^*,y^*)=\sigma^{*(0)}(x^*)\int_0^{y^*}E^{*-1}(y^*)\mathrm{d}y^*+u^{*(0)}(x^*,0)$$

where the constants of integration $\sigma^{*(0)}$ (the zero-order stress) and $u^{*(0)}(x^*, 0)$ are functions of x^* alone. Also, the periodicity can be expressed as:

$$u^{*(0)}(x^*, 1) = u^{*(0)}(x^*, 0)$$

which leads us to:

$$\sigma^{*(0)}(x^*) \int_0^1 E^{*-1}(y^*)\mathrm{d}y^* = 0$$

which means that $\sigma^{*(0)} = 0$, since $E^* > 0$. Finally:

$$u^{*(0)}(x^*, y^*) = u^{*(0)}(x^*)$$

proving that at the dominant order, the displacement is a function of x^* alone. In other words it does not fluctuate over the course of a period.

Second order in ε: the following order gives us:

$$\frac{\partial}{\partial y^*}\left(E^*(y^*)(\frac{\partial u^{*(1)}}{\partial y^*} + \frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*})\right) = 0$$

$$[E^*(y^*)(\frac{\partial u^{*(1)}}{\partial y^*} + \frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*})]_{\Gamma^*} = 0$$

$$[u^{*(1)}]_{\Gamma^*} = 0$$

where $u^{*(1)}$ is 1-periodic in y^*. The general solution to the differential equation can be obtained as before:

$$E^*(y^*)(\frac{\partial u^{*(1)}}{\partial y^*} + \frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*}) = \sigma^{*(1)}(x^*)$$

$$u^{*(1)}(x^*, y^*) = \sigma^{*(1)} \int_0^{y^*} E^{*-1}(y^*)\mathrm{d}y^* - y^* \frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*} + u^{*(1)}(x^*, 0)$$

where $\sigma^{*(1)}$ (first-order stress) and $u^{*(1)}(x^*, 0)$ are functions of x^* alone. Again the periodicity of the unknown $u^{*(1)}(x^*, 1) = u^{*(1)}(x^*, 0)$ allows us to determine $\sigma^{*(1)}$:

$$\sigma^{*(1)}(x^*) \int_0^1 E^{*-1}(y^*)\mathrm{d}y^* - \frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*} = 0$$

$$\sigma^{*(1)}(x^*) = \langle E^{*-1} \rangle^{-1} \frac{\partial u^{*(0)}}{\partial x^*}$$

where $\langle . \rangle$ represents the mean operator over the period, here:

$$\langle . \rangle = \int_0^1 . \, \mathrm{d}y^*$$

Third order in ε^2: Compatibility condition. At this order we have:

$$\frac{\partial}{\partial y^*}\left(E^*(y^*)(\frac{\partial u^{*(2)}}{\partial y^*} + \frac{\partial u^{*(1)}}{\partial x^*})\right) = -\frac{\partial}{\partial x^*}\left(E^*(y^*)(\frac{\partial u^{*(1)}}{\partial y^*} + \frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*})\right) \tag{3.7}$$

$$[E^*(y^*)(\frac{\partial u^{*(2)}}{\partial y^*} + \frac{\partial u^{*(1)}}{\partial x^*})] = 0$$

$$[u^{*(2)}] = 0$$

where $u^{*(2)}$ is 1-periodic in y^*. We do not need to calculate $u^{*(2)}$ as we did for $u^{*(0)}$ and $u^{*(1)}$, at least if we limit ourselves to studying the first macroscopic order. In fact the differential equation represents the conservation of the periodic quantity:

$$\sigma^{*(2)} = E^*(y^*)(\frac{\partial u^{*(2)}}{\partial y^*} + \frac{\partial u^{*(1)}}{\partial x^*})$$

in the presence of the source term:

$$-\frac{\partial}{\partial x^*}\left(E^*(y^*)(\frac{\partial u^{*(1)}}{\partial y^*} + \frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*})\right) = -\frac{\partial \sigma^{*(1)}(x^*)}{\partial x^*}$$

In accordance with the analysis presented in section 2.4.2, equation (3.7) is the exact analog of equation (2.10) with $W^* = 0$. By integrating this conservation equation over the period, we have:

$$\langle \frac{\partial \sigma^{*(2)}}{\partial y^*} \rangle = -\langle \frac{\partial \sigma^{*(1)}(x^*)}{\partial x^*} \rangle$$

But, due to the periodicity, the left hand side is zero:

$$\langle \frac{\partial \sigma^{*(2)}}{\partial y^*} \rangle = \int_0^1 \frac{\partial \sigma^{*(2)}}{\partial y^*} \mathrm{d}y^* = \sigma^{*(2)}(y^* = 1) - \sigma^{*(2)}(y^* = 0)$$

Thus we have established the compatibility condition requiring the source to have a mean of zero (see equation (2.11)):

$$\langle \frac{\partial \sigma^{*(1)}}{\partial x^*} \rangle = 0$$

so that, swapping the derivation with respect to x^* and integration with respect to y^*, and introducing the expression for $\sigma^{*(1)}$:

$$\frac{\mathrm{d}}{\mathrm{d}x^*}\left(\langle E^{*-1}\rangle^{-1}\frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*}\right) = 0 \tag{3.8}$$

This compatibility equation represents, in dimensionless form and to first order of approximation, the equivalent macroscopic description that we were looking for. With:

$$x^* = \frac{X}{L_c}, \qquad E^* = \frac{E}{E_c}, \qquad u^* = \frac{u}{u_c}, \qquad u = u^{(0)} + \bar{\mathcal{O}}(\varepsilon)$$

the stress can be written in dimensional variables:

$$\sigma = \sigma_c \sigma^{*(1)} = (E_c u_c / l_c)\langle E^{*-1}\rangle^{-1}\frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*} = \langle E^{-1}\rangle^{-1}\frac{\mathrm{d}u}{\mathrm{d}X}$$

In the same way, the model can be written in dimensional variables:

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(\langle E^{-1}\rangle^{-1}\frac{\mathrm{d}u}{\mathrm{d}X}\right) = \bar{\mathcal{O}}(\varepsilon)$$

where $\bar{\mathcal{O}}(\varepsilon)$ is a term of relative order ε.

3.3.1.2. *Comments*

3.3.1.2.1. Effective coefficient

The structure of the macroscopic description is identical to that of the local description. In particular, the property $E^* > 0$ is preserved because the macroscopic effective elastic coefficient is such that:

$$E^{\mathrm{eff}*} = \langle E^{*-1}\rangle^{-1} \quad > 0$$

This result is incidentally a classical one, and does not require any particular homogenization technique to prove it (see Chapter 1 where the equivalent thermal problem was treated). We find in this one-dimensional steady state problem that the stress is constant:

$$\sigma = E\frac{\mathrm{d}u}{\mathrm{d}X} = \text{constant}$$

Taking the mean of σ/E over the period, we find:

$$\langle\frac{\sigma}{E}\rangle = \sigma\langle\frac{1}{E}\rangle = \langle\frac{\mathrm{d}u}{\mathrm{d}X}\rangle$$

which leads to the result when it is observed that the mean strain is the macroscopic strain. Finally we note that $E(y^*)$ tends to $\langle E(y^*)\rangle$ when ε tends to zero, weakly in L^2 [SAN 80], but that in general terms:

$$E^{\text{eff}} \neq \langle E(y^*)\rangle$$

3.3.1.2.2. Macroscopic physical quantities

The dimensionless physical quantities – the displacement u^* and the stress σ^* – are given to first order by:

$$u^* = u^{*(0)}(x^*)$$

$$\sigma^* = \varepsilon\sigma^{*(1)} = \varepsilon E^*(y^*)(\frac{\partial u^{*(1)}}{\partial y^*} + \frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*}) = \varepsilon\langle E^{*-1}\rangle^{-1}\frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*}$$

They are independent of the local variable y^* and represent macroscopic quantities, without any mean operator. The physical significance of the macroscopic quantities does not therefore pose any problem here because it is identical to those introduced locally.

3.3.1.2.3. Accuracy of the macroscopic description

Returning to the displacement u^*, the dimensionless macroscopic description (3.8) can be written:

$$\frac{\mathrm{d}}{\mathrm{d}x^*}\left(\langle E^{*-1}\rangle^{-1}\frac{\mathrm{d}u^*}{\mathrm{d}x^*}\right) = \mathcal{O}(\varepsilon)$$

In practice the small parameter ε is non-zero and the equivalent macroscopic description is only approximate. This is the case for any macroscopic description of a heterogenous material.

3.3.1.2.4. Quasi-periodicity: macroscopically heterogenous material

The case of quasi-periodicity where the modulus E^* is not only a function of y^* but also of x^* does not pose any difficulty, as long as the variations are sufficiently slow that a separation of scales is retained. The effective coefficient is still written as $\langle E^{*-1}\rangle^{-1}$, but now it depends on the variable x^*. What happens is that x^* plays the role of a parameter in the process: we recall that the differential systems that must be solved involve the variable y^*. This observation can of course be applied to all homogenization problems, thus making it possible to systematically extend the results to slightly macroscopically heterogenous media.

Finally, when the material is not strictly periodic, in other words when the period Ω^* depends on x^*, (3.8) becomes:

$$\frac{\mathrm{d}}{\mathrm{d}x^*}\left(|\Omega^*|\langle E^{*-1}\rangle^{-1}\frac{\mathrm{d}u^*}{\mathrm{d}x^*}\right) = \mathcal{O}(\varepsilon)$$

3.3.2. *Elasto-dynamics*

We will now include the inertial term. The local description is then given by the system of equations (3.1, 3.2, 3.3). The change is that this system introduces a dimensionless number denoted $\mathcal{P}$, the ratio of the inertial term to the elastic term:

$$\mathcal{P} = \frac{|\rho \dfrac{\partial^2 u}{\partial t^2}|}{|\mathrm{div}_X(E\mathrm{grad}_X u)|}$$

We will again adopt the microscopic viewpoint here, so that the characteristic length for non-dimensionalization is l_c. With:

$$X = l_c y^*, \quad E = E_c E^*, \quad u = u_c u^*, \quad \rho = \rho_c \rho^*, \quad t = t_c t^*$$

it follows in dimensionless form that:

$$\mathrm{div}_{y^*}(E^*\ \mathrm{grad}_{y^*} u^*) = \mathcal{P}_l\ \rho^* \frac{\partial^2 u^*}{\partial t^{*2}} \tag{3.9}$$

$$[\sigma^*]_{\Gamma^*} = [E^*\ \mathrm{grad}_{y^*} u^*]_{\Gamma^*} = 0 \tag{3.10}$$

$$[u^*]_{\Gamma^*} = 0 \tag{3.11}$$

with:

$$\mathcal{P}_l = \frac{\rho_c l_c^2}{E_c t_c^2}$$

Typically the time t_c is linked to the period of the wave, or to its pulsation ω_c by $t_c = 1/\omega_c$. The physical significance of the dimensionless number $\mathcal{P}_l$, the value of $\mathcal{P}$ using l_c as the characteristic length, should be clarified. We can anticipate that the effective elastic modulus E^{eff}, if it exists, is of the order of magnitude of the characteristic modulus E_c. The wave velocity is then:

$$c = \mathcal{O}\left(\sqrt{\frac{E_c}{\rho_c}}\right)$$

and the wavelength λ for pulsations of order ω_c is:

$$\lambda = \mathcal{O}\left(\frac{2\pi}{\omega_c}\sqrt{\frac{E_c}{\rho_c}}\right) = \mathcal{O}\left(2\pi t_c\sqrt{\frac{E_c}{\rho_c}}\right)$$

Finally, $\mathcal{P}_l$ is the squared product of the wavenumber $(2\pi/\lambda)$ and the length of the geometric period:

$$\mathcal{P}_l = \mathcal{O}\left(\left[\frac{2\pi l_c}{\lambda}\right]^2\right) \tag{3.12}$$

We will again look for a displacement u^* of the form:

$$u^*(x^*, y^*, t^*) = u^{*(0)}(x^*, y^*, t^*) + \varepsilon u^{*(1)}(x^*, y^*, t^*) + \cdots \tag{3.13}$$

with $x^* = \varepsilon y^*$, where $\varepsilon = l_c/L_c$ and $u^{*(i)}$ are periodic with respect to the local variable y^*, of period 1.

Before beginning any homogenization, we must evaluate $\mathcal{P}_l$ as function of powers of ε. Different values of $\mathcal{P}_l$ can in fact be imagined, which reveals whether the situation can be homogenized or not. We will begin with the local description which leads to an equivalent macroscopic description of the dynamics. This situation corresponds to a $\mathcal{P}_l = \mathcal{O}(\varepsilon^2)$. Then we will consider values close to $\mathcal{P}_l = \mathcal{O}(\varepsilon^3)$ which lead to a macroscopic description which is steady state to first order of approximation, the case investigated in section 3.2, and finally $\mathcal{P}_l = \mathcal{O}(\varepsilon)$ which corresponds to a local description which cannot be homogenized.

3.3.2.1. *Macroscopic dynamics:* $\mathcal{P}_l = \mathcal{O}(\varepsilon^2)$

3.3.2.1.1. Normalization

We are looking for the local description corresponding to macroscopic dynamics. It must be homogenizable, and so the geometry and disturbance must exhibit a separation of scales. We will assume that this is the case for the geometry. As far as the perturbation goes, $\lambda_c/2\pi$ is a good candidate to define a characteristic macroscopic length L_c, as we will demonstrate in the following section. The separation of scales then requires that:

$$\frac{2\pi l_c}{\lambda} = \varepsilon \ll 1$$

and with (3.12):

$$\mathcal{P}_l = \mathcal{O}\left(\left[\frac{2\pi l_c}{\lambda}\right]^2\right) = \mathcal{O}(\varepsilon^2), \quad \text{so that } \mathcal{P}_l = \varepsilon^2 \mathcal{P}_l^* \text{ with } \mathcal{P}_l^* = \mathcal{O}(1)$$

We can reasonably hope that this estimate of $\mathcal{P}_l$ represents a homogenizable local description which will lead to a macroscopic description of the dynamics. This is proven below. We observe that the condition of separation of scales, in imposing a wavelength which is large relative to l_c, implies as a consequence a frequency ω which must be sufficiently low: $\omega < \omega_{\text{dif}}$ (diffraction becomes significant for frequencies $\mathcal{O}(\omega_{\text{dif}})$ such that λ is of the order of l_c). Equation (3.9) becomes:

$$\text{div}_{y^*}(E^* \,\text{grad}_{y^*} u^*) = \varepsilon^2 \mathcal{P}_l^* \rho^* \frac{\partial^2 u^*}{\partial t^{*2}} \tag{3.14}$$

3.3.2.1.2. Homogenization

Substituting the expansion (3.13) into the dimensionless equations, it is easy to see that the way the first two problems we solved, for unknowns $u^{*(0)}$ and $u^{*(1)}$, are identical to those obtained for the steady state case. We therefore have:

$$u^{*(0)} = u^{*(0)}(x^*, t^*)$$

$$u^{*(1)}(x^*, y^*, t^*) = \sigma^{*(1)}(x^*, t^*) \int_0^{y^*} E^{*-1}(y^*)\mathrm{d}y^* - y^* \frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*} + u^{*(1)}(x^*, 0, t^*)$$

with:

$$\sigma^{*(1)}(x^*, t^*) = \langle E^{*-1} \rangle^{-1} \frac{\partial u^{*(0)}}{\partial x^*}$$

By way of contrast, the next order is modified, with the appearance of the inertial term $-\omega^{*2}\rho^*\mathcal{P}_l^* u^{*(0)}$ in the source term:

$$\frac{\partial}{\partial y^*}\left(E^*(y^*)(\frac{\partial u^{*(2)}}{\partial y^*} + \frac{\partial u^{*(1)}}{\partial x^*})\right)$$

$$= -\frac{\partial}{\partial x^*}\left(E^*(y^*)(\frac{\partial u^{*(1)}}{\partial y^*} + \frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*})\right) + \rho^*\mathcal{P}_l^* \frac{\partial^2 u^{*(0)}}{\partial t^{*2}}$$

$$[E^*(y^*)(\frac{\partial u^{*(2)}}{\partial y^*} + \frac{\partial u^{*(1)}}{\partial x^*})]_{\Gamma^*} = 0$$

$$[u^{*(2)}]_{\Gamma^*} = 0$$

Once again we find an equation analogous to equation (2.10), where W^* is the inertial term. Setting the mean of the source to zero, this leads us to the compatibility condition which gives the macroscopic description:

$$\frac{\mathrm{d}}{\mathrm{d}x^*}\left(\langle E^{*-1}\rangle^{-1} \frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*}\right) = \langle \rho^* \mathcal{P}_l^* \rangle \frac{\partial^2 u^{*(0)}}{\partial t^{*2}} \tag{3.15}$$

in dimensionless form, with:

$$x^* = \frac{X}{L_c}, \;\; E^* = \frac{E}{E_c}, \;\; \omega^* = \frac{\omega}{\omega_c}, \;\; \rho^* = \frac{\rho}{\rho_c}, \;\; u^{*(0)} = \frac{u^{(0)}}{u_c} = \frac{u}{u_c} + \mathcal{O}(\varepsilon)$$

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(\langle E^{-1}\rangle^{-1}\frac{\mathrm{d}u}{\mathrm{d}X}\right) = \frac{E_c t_c^2}{\rho_c L_c^2}\langle \rho \mathcal{P}_l^* \rangle \frac{\partial^2 u}{\partial t^2} + \bar{\mathcal{O}}(\varepsilon)$$

and since:

$$\mathcal{P}_l = \mathcal{O}\left(\left[\frac{2\pi l_c}{\lambda}\right]^2\right) = \varepsilon^2 \mathcal{P}_l^* = \mathcal{O}(\varepsilon^2)$$

we have:

$$\mathcal{P}_L = \mathcal{O}\left(\left[\frac{2\pi L_c}{\lambda}\right]^2\right) = \mathcal{O}(1)$$

It follows that:

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(\langle E^{-1}\rangle^{-1}\frac{\mathrm{d}u}{\mathrm{d}X}\right) = \langle \rho \rangle \frac{\partial^2 u}{\partial t^2} + \bar{\mathcal{O}}(\varepsilon)$$

The return to dimensional variables then occurs without ambiguity. In the next sections, the dimensionless numbers will be taken as equal to their order ε estimate (which is equivalent to taking $\mathcal{P}_l^* = 1$).

3.3.2.1.3. Comments

– The estimate does indeed correspond to a homogenizable situation which leads to a macroscopic description of the dynamics.

– The effective elastic modulus to be used in the dynamic regime is the same as that in the steady state regime!

– The effective density is the mean volume of the local density.

– The dynamic description incorporates the steady state situation as a special case. We just need to set $\omega^* = 0$.

– The macroscopic description is an approximation of order $\mathcal{O}(\varepsilon)$.

– The considerations in section 3.2 about the physical meaning of the macroscopic quantities still apply here.

3.3.2.2. *Steady state:* $\mathcal{P}_l = \mathcal{O}(\varepsilon^3)$

The normalization of equation (3.9) is obvious:

$$\mathrm{div}_{y^*}(E^*\mathrm{grad}_{y^*}u^*) = \varepsilon^3\rho^*\frac{\partial^2 u^*}{\partial t^{*2}}$$

with the relations at the discontinuities remaining unchanged. It is clear that now, up to third order, the problems to be solved are identical to those obtained in section 3.2 for the steady state case. There is now an equivalent macroscopic description given by (3.8):

$$\frac{\mathrm{d}}{\mathrm{d}x^*}\left(\langle E^{*-1}\rangle^{-1}\frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*}\right) = 0$$

As with the other macroscopic descriptions obtained up to now, this is only an approximation. The investigation of the next order (the fourth problem), gives a second approximation of order ε. As can easily be anticipated, this approximation includes an inertial term. For $\mathcal{P}_l = \mathcal{O}(\varepsilon^p)$, $p \geqslant 2$, the dynamics appear at the $(p-2)$th order of approximation.

3.3.2.3. *Non-homogenizable description:* $\mathcal{P}_l = \mathcal{O}(\varepsilon)$

Again the normalization is clear:

$$\mathrm{div}_{y^*}(E^*\ \mathrm{grad}_{y^*}u^*) = \varepsilon\rho^*\frac{\partial^2 u^*}{\partial t^{*2}}$$

with the relations at the discontinuities remaining unchanged. But now only the first problem is the same as that obtained above, with:

$$u^{*(0)} = u^{*(0)}(x^*, t^*)$$

The dynamics appear in the second problem, which can be written:

$$\frac{\partial}{\partial y^*}\left(E^*(y^*)(\frac{\partial u^{*(1)}}{\partial y^*} + \frac{\mathrm{d}u^{*(0)}}{\mathrm{d}x^*})\right) = \rho^*\frac{\partial^2 u^{*(0)}}{\partial t^{*2}}$$

$$[E^*(y)(\frac{\partial u^{*(1)}}{\partial y^*} + \frac{\partial u^{*(0)}}{\partial x^*})]_{\Gamma^*} = 0$$

$$[u^{*(1)}]_{\Gamma^*} = 0$$

where $u^{*(1)}$ is 1-periodic in y^*. The first equation is the conservation of a periodic quantity, and includes the source term $\rho^*\partial^2 u^{*(0)}/\partial t^{*2}$. The compatibility condition implies that this term must have a mean of zero (Fredholm alternative):

$$\langle \rho^* \rangle \frac{\partial^2 u^{*(0)}}{\partial t^{*2}} = 0$$

so, since $\rho^* > 0$:

$$\frac{\partial^2 u^{*(0)}}{\partial t^{*2}} = 0$$

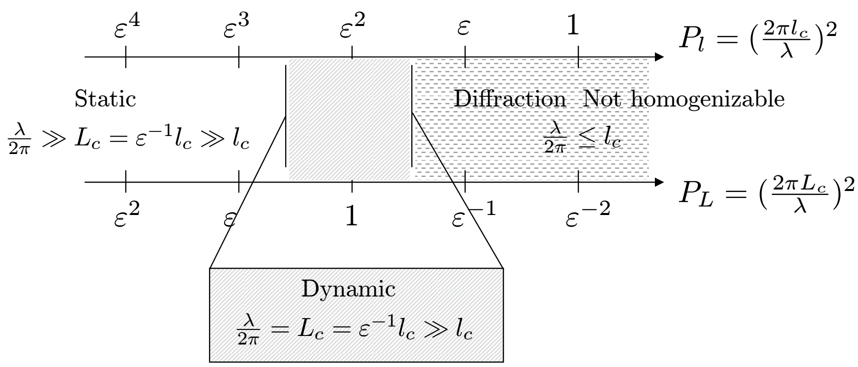


Figure 3.8. *Macroscopic descriptions that may or may not be valid depending on the values of $\mathcal{P}_l$ or $\mathcal{P}_L$*

This result is impossible since $\partial^2 u^{*(0)}/\partial t^{*2}$ is $\mathcal{O}(1)$ by construction. The estimate $\mathcal{P}_l = \mathcal{O}(\varepsilon)$ is a non-homogenizable description. It corresponds to:

$$\mathcal{P}_l = \mathcal{O}\left(\left[\frac{2\pi l}{\lambda}\right]^2\right) = \mathcal{O}(\varepsilon)$$

and hence to:

$$\lambda = \frac{l_c}{\sqrt{\varepsilon}} \ll L_c$$

The dynamic excitation does not fulfill the condition of separation of scales.

To conclude, the different situations are shown in Figure 3.8 as a function of the values of $\mathcal{P}_l$. We observe that the richest macroscopic description, which corresponds to dynamic behavior $\mathcal{P}_l = \mathcal{O}(\varepsilon^2)$, lies at the limit of the homogenizable situations.

3.3.3. *Comments on the different possible choices for the spatial variables*

In order to analyze the previous example we transformed the dimensional spatial variable X into dimensionless spatial variables $x^* = X/L_c$ and $y^* = X/l_c$. In addition, the normalization was carried out by adopting the microscopic viewpoint. The problem was then examined in the space of dimensionless variables, with the return to dimensional variables being carried out at the end of the process.

In the literature, the change into the variables x^* and y^* is often omitted. The treatment is carried out directly in a system of variables x and y, where in general x refers to the normal unit of length, the meter. Alternatively the normalization is carried out by adopting either the micro- or macroscopic viewpoints. The use of these different approaches, although equivalent, is sometimes a source of confusion. It is for this reason that we will now revisit these different methods.

We recall that variables x^* and y^* are particularly well suited to the analysis of problems with a double length scale because, by construction, x^* is the measure of the distance X when using the distance L_c as unit length, and y^* is the measure of the same distance X using the distance $l_c = \varepsilon L_c$ as the unit length. Thus:

– x^* varies by 1 over the macroscopic length L_c (and hence by ε over l_c);

– y^* varies by 1 over the microscopic distance l_c (and hence by ε^{-1} over l_c).

We also note that as a measure of distance in some systems of units, x and y are both dimensionless variables. We will choose x for metric value X, and will designate respectively $\widetilde{L_c}$ and $\widetilde{l_c} = \varepsilon\widetilde{L_c}$ as the metric values of lengths L_c and l_c. Denoting a meter by "1_m", we have the following:

$$X = x^* L_c = x^* \widetilde{L_c}\, 1_m = x\, 1_m$$

and:

$$X = y^* l_c = y^* \widetilde{l_c}\, 1_m = x\, 1_m = y\varepsilon\, 1_m$$

whence it follows that:

$$\frac{x}{x^*} = \frac{y}{y^*} = \widetilde{L_c}$$

which shows that the variables x and y are homothetic to x^* and y^*. We note that here y is a measurement of X in ε m (for example in millimeters for $\varepsilon = 10^{-3}$, etc.). From these we deduce that the derivative operator can take the following equivalent forms:

$$\frac{\partial}{\partial X} = \frac{1}{L_c}\frac{\partial}{\partial x^*} = \frac{1}{1_m}\frac{\partial}{\partial x}$$

or:

$$\frac{\partial}{\partial X} = \frac{1}{l_c}\frac{\partial}{\partial y^*} = \frac{1}{\varepsilon 1_m}\frac{\partial}{\partial y}$$

To illustrate this we will return to the preceding problem in the dynamic regime. We will return to the initial equation, written for convenience in the harmonic regime:

$$\mathrm{div}_X(E\ \mathrm{grad}_X u) = \rho\omega^2 u$$

which can also be written in terms of the variable y^* (microscopic viewpoint):

$$\frac{1}{(\varepsilon L_c)^2}\mathrm{div}_{y^*}(E\ \mathrm{grad}_{y^*} u) = \ \rho\omega^2 u \qquad (3.16)$$

or alternatively, in terms of x^* (macroscopic viewpoint):

$$\frac{1}{L_c^2}\mathrm{div}_{x^*}(E\ \mathrm{grad}_{x^*} u) = \ \rho\omega^2 u \qquad (3.17)$$

Physical analysis showed us that the dynamic regime was characterized by:

$$\mathcal{P}_l = \frac{l_c^2\rho_c\omega_c^2}{E_c} = \mathcal{O}(\varepsilon^2) \qquad \text{or alternatively} \qquad \mathcal{P}_L = \frac{L_c^2\rho_c\omega_c^2}{E_c} = \mathcal{O}(1)$$

Substituting these expressions into (3.17) and (3.16), and changing to the double variable operators, i.e. for the microscopic viewpoint:

$$\frac{\partial}{\partial y^*} \qquad \text{becomes} \qquad \frac{\partial}{\partial y^*} + \varepsilon\frac{\partial}{\partial x^*}$$

and for the macroscopic viewpoint:

$$\frac{\partial}{\partial x^*} \qquad \text{becomes} \qquad \frac{\partial}{\partial x^*} + \varepsilon^{-1}\frac{\partial}{\partial y^*}$$

we obtain the dimensionless formulations established starting with the microscopic viewpoint (already given in the previous section) and the macroscopic viewpoint. We can show that they of course lead to the same equations:

$$(\mathrm{div}_{y^*} + \varepsilon\mathrm{div}_{x^*})\left(E^*(\mathrm{grad}_{y^*} + \varepsilon\mathrm{grad}_{x^*})u^*\right) = \ \varepsilon^2\rho^*\omega^{*2}u^*$$

or:

$$(\mathrm{div}_{x^*} + \varepsilon^{-1}\mathrm{div}_{y^*})\left(E^*(\mathrm{grad}_{x^*} + \varepsilon^{-1}\mathrm{grad}_{y^*})u^*\right) = \ \rho^*\omega^{*2}u^*$$

Also, using the expressions for the derivative operators, we can transform the equations by writing them in terms of the variable y (microscopic viewpoint):

$$\frac{1}{(\varepsilon 1_m)^2}\mathrm{div}_y(E\ \mathrm{grad}_y u) = \rho\omega^2 u$$

or x (macroscopic viewpoint):

$$\frac{1}{{1_m}^2}\mathrm{div}_x(E\ \mathrm{grad}_x u) = \rho\omega^2 u$$

As before, these two equations are normalized in order to describe the dynamic regime. For the mathematical treatment, the unit (1_m) is neutral (since all of the variables and parameters are expressed in the metric system) and so we can abstract ourselves from it. Thus we obtain the formulations in x and y resulting from the microscopic and macroscopic viewpoints:

$$(\mathrm{div}_y + \varepsilon\mathrm{div}_x)\left(E(\mathrm{grad}_y + \varepsilon\mathrm{grad}_x)u\right) = \varepsilon^2\rho\omega^2 u$$

or:

$$(\mathrm{div}_x + \varepsilon^{-1}\mathrm{div}_y)\left(E(\mathrm{grad}_x + \varepsilon^{-1}\mathrm{grad}_y)u\right) = \rho\omega^2 u$$

which, again, are the same. We observe that the use of variables x and y is inconvenient because we lose the unit variation over the micro- or macroscopic distances. The advantage is we can continue to use the normal system of units (metric), and maintain the dimensional physical parameters throughout the treatment. At the end of the process all that needs to be done is to restore the meter as the unit. In other words, replace the value x with the distance X in order to obtain the dimensional formulation. We also note that the equations in x^*, y^* or x, y are formally identical and lead to an identical treatment.

As a final example, consider the quasi-static case corresponding to:

$$\mathcal{P}_l = \frac{l_c^2\rho_c\omega_c^2}{E_c} = \mathcal{O}(\varepsilon^3) \qquad \text{or alternatively} \qquad \mathcal{P}_L = \frac{L_c^2\rho_c\omega_c^2}{E_c} = \mathcal{O}(\varepsilon)$$

The normalizations are, in terms of the variable y^* (microscopic viewpoint):

$$\frac{1}{(\varepsilon L_c)^2}\mathrm{div}_{y^*}(E\ \mathrm{grad}_{y^*} u) = \varepsilon\rho\omega^2 u$$

and, in terms of the variable x^* (macroscopic viewpoint):

$$\frac{1}{L_c^2}\mathrm{div}_{x^*}(E\ \mathrm{grad}_{x^*} u) = \varepsilon\rho\omega^2 u$$

Transforming the derivative operators, we find in terms of the variable y (microscopic viewpoint), and after canceling out the meter units:

$$\text{div}_y(E\ \text{grad}_y u) = \varepsilon^3 \rho\omega^2 u$$

and, in terms of the variable x (macroscopic viewpoint):

$$\text{div}_x(E\ \text{grad}_x u) = \varepsilon \rho\omega^2 u$$

After introducing derivative operators for the double variables, we again reach the same conclusions about the equivalence of the different approaches.

3.4. Expressing problems within the formalism of multiple scales

The above example shows the general approach to be taken in order to establish various behaviors depending on the assumptions made. However, when a macroscopic description is sought for a real phenomenon within a given material, one of the difficulties is that of expressing the assumptions within the formalism of homogenization, in accordance with the problem under investigation. In the previous example, the question would be the following: if a material (for example a soft rock) has the following characteristic values: $l_c = 1$ mm, $E_c = 8 \times 10^9$ Pa, $\rho_c = 2 \times 10^3$ kg/m^3, and cycles of testing at a frequency of 3 kHz are performed on a lattice of size $H = 10$ cm, which of the models that we obtained is the appropriate one to use?

3.4.1. *How do we select the correct mathematical formulation based on the problem at hand?*

The macroscopic description will only be valid if the physics at the microscopic scale is described correctly. The physical analysis of the problem is thus a crucial stage that must occur before the process of homogenization. We have seen that dimensional analysis is an extremely useful tool for carrying out this process correctly. The problem is expressed in dimensionless form and, in order to correctly account for the importance of each term, the dimensionless numbers are expressed in powers of ε. This normalization phase is a key point in the process because that is where the physics of the phenomenon is taken into account. We emphasize that normalizing the dimensionless numbers in terms of powers of ε ensures that the various physical effects are accounted for to the same order, independent of the value of $\varepsilon \ll 1$. Thus when a description is normalized it retains the nature of the physics that applies to the situation, but does not contain any reference to the effective value of ε which, although small, is still not zero.

Nevertheless, it is rare that the normalization follows naturally from the problem under consideration. In particular, when several small parameters are involved (ratios of properties, characteristic times, etc.), several possibilities are available and one should be chosen which applies to the situation being examined. Examples include bituminous concretes, whose behavior varies strongly with temperature and frequency of the load [BOU 89b; BOU 90] (see Chapter 9), or cement pastes which change from a fluid to solid state when they set [BOU 95]. We will show later that this difficulty can be overcome by analyzing the value taken by the scale ratio ε_r in the actual problem.

This idea is clear in the previous example where the choice of model depends on the value of $\mathcal{P}_l$ as a function of ε. For the problem in question, with the numerical values given above, $\mathcal{P}_l$ can be estimated objectively:

$$\mathcal{P}_l = \frac{\rho_c l_c^2}{E_c t_c^2} = \frac{2\ 10^3 (10^{-3})^2}{8\ 10^9 (2\pi\ 3\ 10^3)^{-2}} \simeq 10^{-4}$$

However ε is not specified. We should also point out that if we assume ε to be infinitely small, this is equivalent to considering $\mathcal{P}_l = \mathcal{O}(1)$, which is a situation that cannot be homogenized! Also, considering an arbitrary value of ε to give a scale to $\mathcal{P}_l$ is equivalent to making an arbitrary choice in the constitutive model. To avoid this impasse we are therefore forced to come up with a realistic estimate of ε for the problem being considered.

3.4.2. *Need to evaluate the actual scale ratio $\boldsymbol{\varepsilon_r}$*

The difficulty here is the gulf between:

– the mathematical view, where $\varepsilon = l_c/L_c \rightarrow 0$ is infinitely small and the macroscopic description in this limit is infinitely accurate, corresponding to heterogenities which are infinitely small compared to the macroscopic scale, or alternatively to macroscopic dimensions which are infinitely large compared to the heterogenities;

– the physical reality where this ideal situation is not reached because the size of the REV is finite ($l_c \neq 0$) and the macroscopic scale is not infinite ($L_c \neq \infty$) so that the *actual* scale ratio takes a value which is small but non-zero ($0 < \varepsilon_r \ll 1$).

We can reconcile these two viewpoints by evaluating ε_r. Indeed if ε_r can be estimated, the dimensionless numbers of the real problem can be evaluated in terms of powers of ε_r. Thus we can define a normalization which is consistent with the physics of the problem. If, with this normalization, we carry out homogenization, we obtain a macroscopic description in which all the physical mechanisms act with the same strengths as in the actual problem. Because of this, the problem being considered is only an imperfect example (for $\varepsilon = \varepsilon_r$) of the macroscopic description we have developed, with the discrepancy being smaller when ε_r is small, in other words when the separation of scales is clear. In this case, the zero-order description matches the actual behavior up to order $\mathcal{O}(\varepsilon_r)$.

To summarize, there are two reasons we need to evaluate ε_r: the correct description of the local physics and the estimation of error in the macroscopic description.

3.4.3. *Evaluation of the actual scale ratio $\varepsilon_\mathbf{r}$*

For a given problem, l_c is known for the medium, but the characteristic macroscopic size L_c which, as we have just seen, is crucial for selection of the correct model, is one of the unknowns. The literature is still rather unclear on this issue: this dimension is often associated with the size of the medium under study but, depending on the problem of interest, L_c might alternatively depend on the boundary conditions imposed or on a characteristic dimension of phenomenon such as a wavelength, or a thickness of a viscous layer, etc. In order to evaluate L_c (and ε_r), we will follow the approach proposed by [BOU 89b; 89a] which consists of observing that the process of homogenization must necessarily lead to a quantity – in the case considered above, the displacement $u^{(0)}$ – with the following dimensional form:

$$u^{(0)}(X) + \varepsilon u^{(1)}(X, \varepsilon^{-1}X) + ... \qquad \text{with} \qquad \mathcal{O}(u^{(0)}) = \mathcal{O}(u^{(1)})$$

Turning the problem around, we can say that results of the homogenization will only be applicable to the real situation if this (necessary) condition is satisfied when ε takes the value ε_r. In other words if the variations in $u^{(0)}$ are effectively negligible (i.e. $\mathcal{O}(\varepsilon_r)$) over the period. If we consider for example the growth of $u^{(0)}$ over a period in the direction X_1, we must therefore necessarily have:

$$|u^{(0)}(X_1 + l_c) - u^{(0)}(X_1)| \leqslant \mathcal{O}(\varepsilon_r |u^{(0)}(X_1)|)$$

On the macroscopic scale l_c is very small and we can write:

$$l_c. \left| \frac{\partial u^{(0)}}{\partial X_1} \right| |u^{(0)}|^{-1} \leqslant \mathcal{O}(\varepsilon_r)$$

This gives an underestimation of ε_r. However since ε_r is a measure of macroscopic accuracy, the optimum value is the smallest one that is permissible, which means we can write:

$$\varepsilon_r \simeq l_c \frac{|\frac{\partial u^{(0)}}{\partial X_1}|}{|u^{(0)}|} \qquad \text{or} \qquad L_c \simeq \frac{|u^{(0)}|}{|\frac{\partial u^{(0)}}{\partial X_1}|} \tag{3.18}$$

In the general case where the displacement field is three-dimensional, we have:

$$\varepsilon_r \simeq l_c \max\left(\frac{|\frac{\partial u_i^{(0)}}{\partial X_j}|}{|u_j^{(0)}|}\right) \qquad \text{and} \qquad L_c \simeq \min\left(\frac{|u_i^{(0)}|}{|\frac{\partial u_i^{(0)}}{\partial X_j}|}\right)$$

Locally, an order of magnitude for ε_r is thus given by relative variation of the displacement field over a period. This is equivalent to the estimate that would be obtained by dimensional analysis carried out directly on the macroscopic scale: the slower (or faster) the spatial variations in $u^{(0)}$ the larger (or smaller) L_c is, and the "smaller" (larger) ε_r is (in other words the accuracy is greater (or smaller)). We note that ε_r depends on the geometry of the field, and because of this it is not generally constant in the material, but can vary depending on load, boundary conditions, etc. Our estimate of (3.18) can answer the questions of accuracy and validity of the zero-order macroscopic description.

We will now give an evaluation of ε_r in several familiar situations.

3.4.3.1. *Homogenous treatment of simple compression*

The displacement in a sample of height H takes the form (Figure 3.9): $u^{(0)} = aX$, from which it follows that $\mathcal{O}(u^{(0)}) = aH$ and $\frac{\partial u^{(0)}}{\partial X} = a$, so that:

$$L_c = \frac{|u^{(0)}|}{|\frac{\partial u^{(0)}}{\partial X}|} = \frac{aH}{a} = H \qquad \text{and} \qquad \varepsilon_r = \frac{l_c}{H}$$

It follows that an accuracy of order 10% for the constitutive law requires samples with dimensions which are around 10 times larger than the size of the heterogenities.

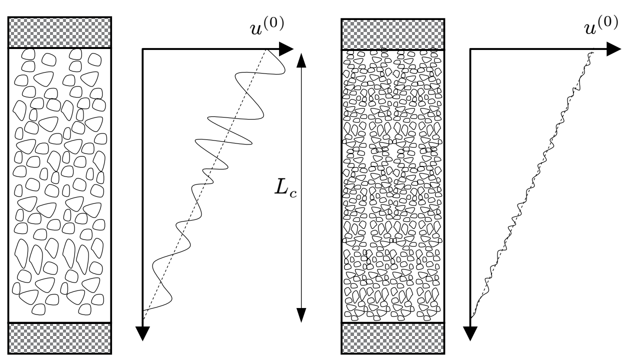


Figure 3.9. *Estimate of the physical scale ratio ε_r in the case of simple compression: $\varepsilon_r = \mathcal{O}(l_c/L_c)$*

3.4.3.2. *Point force in an elastic object*

This is a case where the value of ε_r is not constant in the medium. In fact the displacement field varies as $\sim 1/r^2$ (Figure 3.10), which gives:

$$L_c \simeq \frac{|u^{(0)}|}{|\frac{\partial u^{(0)}}{\partial X}|} = \frac{r}{2} \qquad \text{and} \qquad \varepsilon_r = 2\frac{l_c}{r}$$

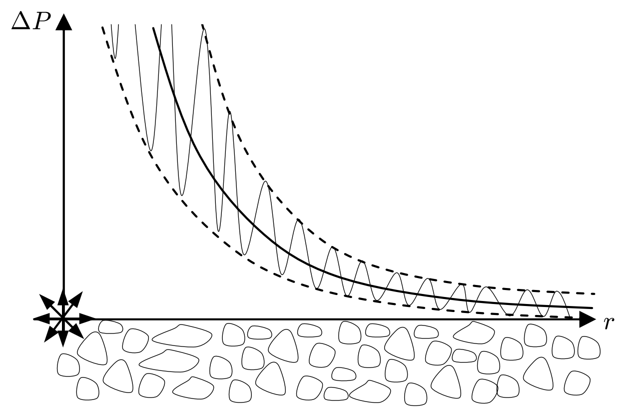


Figure 3.10. *Estimate of the physical scale ratio ε_r for a point source in a porous medium: $\varepsilon_r = 2l_c/r$*

where r is the distance to the point force. From this we can deduce that close to the point force is applied, the phenomenon is not homogenizable. Taking into account the effects of the microstructure, it becomes homogenizable beyond a radius $R \approx 10l_c$. The simple continuum description becomes acceptable at distances greater than $200l_c$ (with an accuracy in the order of a few percent).

3.4.3.3. *Propagation of a harmonic plane wave in elastic composites*

The displacement created by a plane wave in an infinite medium (Figure 3.11) has the form:

$$u^{(0)}(X,t) = |u^{(0)}| \exp(2i\pi(t/T - X/\lambda))$$

and consequently:

$$\frac{\partial u^{(0)}}{\partial X} = -(2i\pi/\lambda)|u^{(0)}| \exp(2i\pi(t/T - X/\lambda))$$

whence:

$$L_c = \frac{|u^{(0)}|}{|\frac{\partial u^{(0)}}{\partial X}|} = \frac{\lambda}{2\pi} \qquad \text{and} \qquad \varepsilon_r = \frac{2\pi l_c}{\lambda}$$

Again we find that the closer we get to the diffraction regime, the poorer the zero-order description performs, so that we require higher order corrections [BOU 96b; AUR 05a]. For wavelengths shorter that $2\pi l_c$, homogenization is no longer applicable.

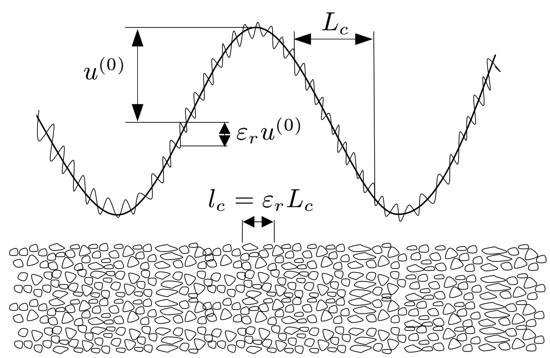


Figure 3.11. *Estimate of the physical scale ratio ε_r for wave propagation:* $\varepsilon_r = 2\pi l_c/\lambda$

3.4.3.4. *Diffusion wave in heterogenous media*

For a harmonic plane wave of thermal diffusion, temperature takes the form:

$$\theta^{(0)}(X,t) = |\theta^{(0)}| \exp(2i\pi(t/T - X\sqrt{i}/\delta_t))$$

where δ is the wavelength of thermal diffusion:

$$\delta_t = \sqrt{\frac{\lambda}{\rho C \omega}}$$

λ is thermal conductivity and ρC is heat capacity. As a result:

$$\frac{\partial \theta^{(0)}}{\partial X} = -(2\sqrt{i}\pi/\delta_t)|\theta^{(0)}| \exp(2i\pi(t/T - X\sqrt{i}/\delta_t))$$

so that:

$$L_c = \frac{|\theta^{(0)}|}{|\frac{\partial \theta^{(0)}}{\partial X}|} = \frac{\delta_t}{2\pi} \qquad \text{and} \qquad \varepsilon_r = \frac{2\pi l_c}{\delta_t}$$

The same applies for diffusive waves as for elastic waves, and it is the wavelength which determines the macroscopic scale.

3.4.3.5. *Conclusions to be drawn from the examples*

Let us return to the example of dynamic measurements of a rock sample. At a frequency of 3 kHz we have the following estimate of the wavelength:

$$\frac{\lambda}{2\pi} = \frac{c}{\omega} \simeq \sqrt{\frac{E_c}{\rho_c}}\frac{1}{\omega} = \sqrt{\frac{8.10^9}{2.10^3}}\frac{1}{2\pi 3.10^3} \simeq 0.1\mathrm{m}$$

which corresponds here to the height H of the sample. The value of ε_r is thus (with $l_c = 1$ mm):

$$\varepsilon_r = \frac{2\pi l_c}{\lambda} = \frac{l_c}{H} = 10^{-2}$$

so that, following the value of $\mathcal{P}_l$ estimated above:

$$\mathcal{P}_l = 10^{-4} = \varepsilon_r^2$$

It is thus natural that we should use the model corresponding to $\mathcal{P}_l = \mathcal{O}(\varepsilon^2)$, in other words the dynamic description. If, for the same material, tests were carried out at $f' = 300$ Hz, we would have:

$$\mathcal{P'}_l = \frac{\rho_c l_c^2}{E_c t_c^2} = \frac{2\ 10^3(10^{-3})^2}{8\ 10^9(2\pi 3\ 10^2)^{-2}} \simeq 10^{-6} \qquad \text{and} \qquad \frac{\lambda'}{2\pi} \simeq 1\mathrm{m} > H$$

In this case the macroscopic size is no longer defined by the wavelength but by the dimensions of the sample, and this time we have:

$$\varepsilon'_r = \frac{l_c}{H} = 10^{-2} \qquad \text{so that:} \qquad \mathcal{P'}_l = 10^{-6} = \varepsilon_r^3$$

which leads us to use the model for $\mathcal{P}_l = \mathcal{O}(\varepsilon^3)$, in other words the quasi-static description. Conversely, tests carried out at 30 kHz give $\mathcal{P}_l \simeq 10^{-2}$, $\lambda/2\pi \simeq 1$ cm $< H$, and $\varepsilon_r = 10^{-1}$, so that $\mathcal{P}_l = \mathcal{O}(\sqrt{\varepsilon_r})$, putting the tests in the dynamic regime, at the limit of what is homogenizable. As for the diffractive regime where homogenization is no longer applicable due to the absence of a separation of scales, this is reached at frequencies where $(\lambda/2\pi) \simeq l_c$ so that $\varepsilon_r \simeq 1$.