Table of Contents

Introduction. Qualitative Reasoning xvii

Chapter 1. Allen's Calculus 1

1.1. Introduction 1
1.1.1. "The mystery of the dark room" 1
1.1.2. Contributions of Allen's formalism 5
1.2. Allen's interval relations 6
1.2.1. Basic relations 6
1.2.2. Disjunctive relations 7
1.3. Constraint networks 8
1.3.1. Definition 8
1.3.2. Expressiveness 10
1.3.3. Consistency 13
1.4. Constraint propagation 17
1.4.1. Operations: inversion and composition 17
1.4.2. Composition table 18
1.4.3. Allen's algebra 20
1.4.4. Algebraic closure 20
1.4.5. Enforcing algebraic closure 22
1.5. Consistency tests 26
1.5.1. The case of atomic networks 26
1.5.2. Arbitrary networks 27
1.5.3. Determining polynomial subsets 28

Chapter 2. Polynomial Subclasses of Allen's Algebra 29

2.1. "Show me a tractable relation!" 29
2.2. Subclasses of Allen's algebra 30
2.2.1. A geometrical representation of Allen's relations 30
2.2.2. Interpretation in terms of granularity 33

2.2.3. Convex and pre-convex relations 35
2.2.4. The lattice of Allen's basic relations 38
2.2.5. Tractability of convex relations 44
2.2.6. Pre-convex relations 45
2.2.7. Polynomiality of pre-convex relations 47
2.2.8. ORD-Horn relations 51
2.3. Maximal tractable subclasses of Allen's algebra 52
2.3.1. An alternative characterization of pre-convex relations 52
2.3.2. The other maximal polynomial subclasses 54
2.4. Using polynomial subclasses 57
2.4.1. Ladkin and Reinefeld's algorithm 57
2.4.2. Empirical study of the consistency problem 59
2.5. Models of Allen's language 60
2.5.1. Representations of Allen's algebra 60
2.5.2. Representations of the time-point algebra 60
2.5.3. $\aleph_0$-categoricity of Allen's algebra 61
2.6. Historical note 61

Chapter 3. Generalized Intervals 63
3.1. "When they built the bridge . . . " 63
3.1.1. Towards generalized intervals 64
3.2. Entities and relations 65
3.3. The lattice of basic (p, q)-relations 68
3.4. Regions associated with basic (p, q)-relations 69
3.4.1. Associated polytopes 71
3.4.2. M-convexity of the basic relations 73
3.5. Inversion and composition 73
3.5.1. Inversion 73
3.5.2. Composition 74
3.5.3. The algebras of generalized intervals 77
3.6. Subclasses of relations: convex and pre-convex relations 79
3.6.1. (p, q)-relations 79
3.6.2. Convex relations 79
3.6.3. Pre-convex relations 81
3.7. Constraint networks 82
3.8. Tractability of strongly pre-convex relations 83
3.8.1. ORD-Horn relations 84
3.9. Conclusions 84
3.10. Historical note 85

Chapter 4. Binary Qualitative Formalisms 87
4.1. "Night driving" 87
4.1.1. Parameters 89
4.1.2. A panorama of the presented formalisms 89

4.2. Directed points in dimension 1 92
4.2.1. Operations 93
4.2.2. Constraint networks 93
4.2.3. Networks reducible to point networks 94
4.2.4. Arbitrary directed point networks 97
4.3. Directed intervals 97
4.3.1. Operations 98
4.3.2. Constraint networks and complexity 98
4.4. The $\mathcal{OPRA}$ direction calculi 99
4.5. Dipole calculi 100
4.6. The Cardinal direction calculus 101
4.6.1. Convex and pre-convex relations 102
4.6.2. Complexity 103
4.7. The Rectangle calculus 104
4.7.1. Convex and pre-convex relations 105
4.7.2. Complexity 106
4.8. The n-point calculus 106
4.8.1. Convexity and pre-convexity 107
4.9. The n-block calculus 108
4.9.1. Convexity and pre-convexity 108
4.10. Cardinal directions between regions 109
4.10.1. Basic relations 109
4.10.2. Operations 111
4.10.3. Consistency of basic networks 112
4.10.4. Applications of the algorithm 122
4.11. The INDU calculus 123
4.11.1. Inversion and composition 123
4.11.2. The lattice of *INDU* relations 123
4.11.3. Regions associated with *INDU* relations 124
4.11.4. A non-associative algebra 125
4.12. The $2n$-star calculi 126
4.12.1. Inversion and composition 127
4.13. The Cyclic interval calculus 128
4.13.1. Convex and pre-convex relations 130
4.13.2. Complexity of the consistency problem 131
4.14. The RCC–8 formalism 131
4.14.1. Basic relations 132
4.14.2. Allen's relations and RCC-8 relations 133
4.14.3. Operations 134
4.14.4. Maximal polynomial classes of RCC-8 135
4.15. A discrete RCC theory 137
4.15.1. Introduction 137
4.15.2. Entities and relations 137

4.15.3. Mereology 138
4.15.4. Concept of contact and RCC−8 relations 138
4.15.5. Closure, interior and boundary 139
4.15.6. Self-connectedness 140
4.15.7. Paths, distance, and arc-connectedness 141
4.15.8. Distance between regions 142
4.15.9. Conceptual neighborhoods 143

Chapter 5. Qualitative Formalisms of Arity Greater than 2 145

5.1. "The sushi bar" 145
5.2. Ternary spatial and temporal formalisms 146
5.2.1. General concepts 146
5.2.2. The Cyclic point calculus 147
5.2.3. The Double-cross calculus 148
5.2.4. The Flip-flop and $\mathcal{LR}$ calculi 151
5.2.5. Practical and natural calculi 152
5.2.6. The consistency problem 153
5.3. Alignment relations between regions 155
5.3.1. Alignment between regions of the plane: the 5-intersection calculus 155
5.3.2. Ternary relations between solids in space 156
5.3.3. Ternary relations on the sphere 157
5.4. Conclusions 158

Chapter 6. Quantitative Formalisms, Hybrids, and Granularity 159

6.1. "Did John meet Fred this morning?" 159
6.1.1. Contents of the chapter 160
6.2. TCSP metric networks 160
6.2.1. Operations 161
6.2.2. The consistency problem 162
6.3. Hybrid networks 164
6.3.1. Kautz and Ladkin's formalism 164
6.4. Meiri's formalism 168
6.4.1. Temporal entities and relations 169
6.4.2. Constraint networks 170
6.4.3. Constraint propagation 170
6.4.4. Tractability issues 172
6.5. Disjunctive linear relations (DLR) 174
6.5.1. A unifying formalism 174
6.5.2. Allen's algebra with constraints on durations 174
6.5.3. Conclusions 175
6.6. Generalized temporal networks 175
6.6.1. Motivations 176

6.6.2. Definition of GTN 176
6.6.3. Expressiveness 177
6.6.4. Constraint propagation 178
6.6.5. Conclusions 179
6.7. Networks with granularity 179
6.7.1. Introduction 179
6.7.2. Granularities and granularity systems 180
6.7.3. Constraint networks 182
6.7.4. Complexity of the consistency problem 184
6.7.5. Propagation algorithms 184

Chapter 7. Fuzzy Reasoning 187

7.1. "Picasso's Blue period" 187
7.2. Fuzzy relations between classical intervals 188
7.2.1. Motivations 188
7.2.2. The fuzzy Point algebra 189
7.2.3. The fuzzy Interval algebra 191
7.2.4. Fuzzy constraint networks 192
7.2.5. Algorithms, tractable subclasses 193
7.2.6. Assessment 195
7.3. Events and fuzzy intervals 195
7.3.1. Fuzzy intervals and fuzzy relations 195
7.3.2. Fuzzy constraints 197
7.3.3. An example of application 202
7.3.4. Complexity 204
7.3.5. Weak logical consequence 206
7.3.6. Assessment 208
7.4. Fuzzy spatial reasoning: a fuzzy RCC 208
7.4.1. Motivations 208
7.4.2. Fuzzy regions 209
7.4.3. Fuzzy RCC relations 209
7.4.4. Fuzzy RCC formulas 212
7.4.5. Semantics 212
7.4.6. Satisfying a finite set of normalized formulas 213
7.4.7. $(n; \alpha, \beta)$-models 215
7.4.8. Satisfiability and linear programming 217
7.4.9. Models with a finite number of degrees 218
7.4.10. Links with the egg-yolk calculus 218
7.5. Historical note 222

Chapter 8. The Geometrical Approach and Conceptual Spaces 223

8.1. "What color is the chameleon?" 223
8.2. Qualitative semantics 224

8.3. Why introduce topology and geometry? . . . 225
8.4. Conceptual spaces . . . 226
8.4.1. Higher order properties and relations . . . 228
8.4.2. Notions of convexity . . . 228
8.4.3. Conceptual spaces associated to generalized intervals . . . 230
8.4.4. The conceptual space associated to directed intervals . . . 230
8.4.5. Conceptual space associated with cyclic intervals . . . 231
8.4.6. Conceptual neighborhoods in Allen's relations . . . 234
8.4.7. Dominance spaces and dominance diagrams . . . 235
8.5. Polynomial relations of INDU . . . 237
8.5.1. Consistency . . . 238
8.5.2. Convexity and Horn clauses . . . 246
8.5.3. Pre-convex relations . . . 247
8.5.4. NP-completeness of pre-convex relations . . . 248
8.5.5. Strongly pre-convex relations . . . 248
8.5.6. The subclass $\mathcal{G}$ . . . 252
8.5.7. A summary of complexity results for INDU . . . 257
8.6. Historical note . . . 258

Chapter 9. Weak Representations . . . 259

9.1. "Find the hidden similarity" . . . 259
9.2. Weak representations . . . 261
9.2.1. Weak representations of the point algebra . . . 261
9.2.2. Weak representations of Allen's interval algebra . . . 262
9.2.3. Weak representations of the n-interval algebra . . . 271
9.2.4. Constructing the canonical configuration . . . 273
9.3. Classifying the weak representations of $\mathbf{A}_n$ . . . 275
9.3.1. The category of weak representations of $\mathbf{A}_n$ . . . 275
9.3.2. Reinterpretating the canonical construction . . . 277
9.3.3. The canonical construction as adjunction . . . 279
9.4. Extension to the calculi based on linear orders . . . 283
9.4.1. Configurations . . . 284
9.4.2. Description languages and associated algebras . . . 284
9.4.3. Canonical constructions . . . 285
9.4.4. The construction in the case of $\mathbf{A}_{Points_n}$ . . . 288
9.5. Weak representations and configurations . . . 290
9.5.1. Other qualitative formalisms . . . 290
9.5.2. A non-associative algebra: INDU . . . 291
9.5.3. Interpreting Allen's calculus on the integers . . . 291
9.5.4. Algebraically closed but inconsistent scenarios: the case of cyclic intervals . . . 291
9.5.5. Weak representations of RCC−8 . . . 292
9.5.6. From weak representations to configurations . . . 302

9.5.7. Finite topological models 302
9.5.8. Models in Euclidean space 303
9.6. Historical note 304

Chapter 10. Models of RCC−8 305

10.1. "Disks in the plane" 305
10.2. Models of a composition table 307
10.2.1. Complements on weak representations 307
10.2.2. Properties of weak representations 308
10.2.3. Models of the composition table of RCC−8 311
10.3. The RCC theory and its models 312
10.3.1. Composition tables relative to a logical theory 312
10.3.2. The RCC theory 313
10.3.3. Strict models and Boolean connection algebras 315
10.3.4. Consistency of strict models 318
10.4. Extensional entries of the composition table 319
10.4.1. Properties of the triads of a composition table 320
10.4.2. Topological models of RCC 322
10.4.3. Pseudocomplemented lattices 323
10.4.4. Pseudocomplementation and connection 326
10.4.5. Non-strict models 326
10.4.6. Models based on regular closed sets 328
10.5. The generalized RCC theory 329
10.5.1. The mereological component: variations of a set-theoretic theme 330
10.5.2. The topological component 333
10.5.3. The GRCC theory 335
10.5.4. Constructing generalized Boolean connection algebras 335
10.5.5. An application to finite models 336
10.6. A countable connection algebra 337
10.6.1. An interval algebra 337
10.6.2. Defining a connection relation 338
10.6.3. Minimality of the algebra (B_ω, C_ω) 341
10.7. Conclusions 341

Chapter 11. A Categorical Approach of Qualitative Reasoning 343

11.1. "Waiting in line" 343
11.2. A general construction of qualitative formalisms 346
11.2.1. Partition schemes 346
11.2.2. Description of configurations 347
11.2.3. Weak composition 347
11.2.4. Weak composition and seriality 348
11.3. Examples of partition schemes 349

11.4. Algebras associated with qualitative formalisms 350
11.4.1. Algebras associated with partition schemes 350
11.4.2. Examples 351
11.4.3. Associativity 351
11.5. Partition schemes and weak representations 352
11.5.1. The weak representation associated with a partition scheme . . . 353
11.6. A general definition of qualitative formalisms 353
11.6.1. The pivotal role of weak representations 354
11.6.2. Weak representations as constraints 354
11.6.3. Weak representations as interpretations 355
11.7. Interpretating consistency 355
11.7.1. Inconsistent weak representations 356
11.8. The category of weak representations 357
11.8.1. The algebra of partial orders 357
11.8.2. On the relativity of consistency 358
11.8.3. Semi-strong weak representations 359
11.9. Conclusions 360

Chapter 12. Complexity of Constraint Languages 363

12.1. "Sudoku puzzles" 363
12.2. Structure of the chapter 365
12.3. Constraint languages 366
12.4. An algebraic approach of complexity 367
12.5. CSPs and morphisms of relational structures 368
12.5.1. Basic facts about CSPs 368
12.5.2. CSPs and morphisms of structures 369
12.5.3. Polymorphisms and invariant relations 370
12.5.4. pp-Definable relations and preservation 371
12.5.5. A Galois correspondence 372
12.6. Clones of operations 373
12.6.1. The Boolean case 374
12.7. From local consistency to global consistency 375
12.8. The infinite case 376
12.8.1. Homogeneous and $\aleph_0$-categorical structures 376
12.8.2. Quasi near-unanimity operations 377
12.8.3. Application to the point algebra 378
12.8.4. Application to the RCC−5 calculus 379
12.8.5. Application to pointizable relations of Allen's algebra 380
12.8.6. Applications to the study of complexity 381
12.9. Disjunctive constraints and refinements 382
12.9.1. Disjunctive constraints 382
12.9.2. Guaranteed satisfaction property 383

12.9.3. The k-independence property 384
12.9.4. Refinements 385
12.10. Refinements and independence 389
12.11. Historical note 390

Chapter 13. Spatial Reasoning and Modal Logic 391

13.1. "The blind men and the elephant" 391
13.2. Space and modal logics 393
13.3. The modal logic S4 393
13.3.1. Language and axioms 393
13.3.2. Kripke models 394
13.4. Topological models 396
13.4.1. Topological games 397
13.4.2. The rules of the game 397
13.4.3. End of game 398
13.4.4. Examples of games: example 1 398
13.4.5. Example 2 399
13.4.6. Example 3 400
13.4.7. Games and modal rank of formulas 401
13.4.8. McKinsey and Tarski's theorem 402
13.4.9. Topological bisimulations 404
13.4.10. Expressiveness 405
13.4.11. Relational and topological models 405
13.4.12. From topological models to Kripke models 406
13.4.13. An extended language: $\mathrm{S4}_u$ 408
13.5. Translating the RCC−8 predicates 408
13.6. An alternative modal translation of RCC−8 409
13.7. Generalized frames 410
13.8. Complexity 411
13.9. Complements 412
13.9.1. Analogs of Kamp operators 412
13.9.2. Space and epistemic logics 412
13.9.3. Other extensions 412

Chapter 14. Applications and Software Tools 413

14.1. Applications 413
14.1.1. Applications of qualitative temporal reasoning 413
14.1.2. Applications of spatial qualitative reasoning 414
14.1.3. Spatio-temporal applications 415
14.2. Software tools 416
14.2.1. The Qualitative Algebra Toolkit (QAT) 416
14.2.2. The SparQ toolbox 417
14.2.3. The GQR system 419

Chapter 15. Conclusion and Prospects 423

15.1. Introduction 423
15.2. Combining qualitative formalisms 423
15.2.1. Tight or loose integration 424
15.2.2. RCC−8 and the Cardinal direction calculus 425
15.2.3. RCC−8, the Rectangle calculus, and the Cardinal direction calculus between regions 425
15.2.4. The lattice of partition schemes 426
15.3. Spatio-temporal reasoning 426
15.3.1. Trajectory calculi 426
15.3.2. Reasoning about space-time 428
15.3.3. Combining space and time 430
15.4. Alternatives to qualitative reasoning 430
15.4.1. Translation in terms of finite CSP 431
15.4.2. Translation into an instance of the SAT problem 433
15.5. To conclude — for good 434

Appendix A. Elements of Topology 435

A.1. Topological spaces 435
A.1.1. Interior, exterior, boundary 436
A.1.2. Properties of the closure operator 436
A.1.3. Properties of the interior operator 437
A.1.4. Closure, interior, complement 437
A.1.5. Defining topological spaces in terms of closed sets 438
A.1.6. Defining topological spaces in terms of operators 438
A.1.7. Pseudocomplement of an open set 439
A.1.8. Pseudosupplement of a closed set 439
A.1.9. Regular sets 440
A.1.10. Axioms of separation 440
A.1.11. Bases of a topology 440
A.1.12. Hierarchy of topologies 441
A.1.13. Preorder topology 441
A.1.14. Constructing topological spaces 442
A.1.15. Continuous maps 443
A.1.16. Homeomorphisms 444
A.2. Metric spaces 445
A.2.1. Minkowski metrics 445
A.2.2. Balls and spheres 445
A.2.3. Bounded sets 446
A.2.4. Topology of a metric space 446
A.2.5. Equivalent metrics 446

A.2.6. Distances between subsets 447
A.2.7. Convergence of a sequence 447
A.3. Connectedness and convexity 447
A.3.1. Connectedness 447
A.3.2. Connected components 448
A.3.3. Convexity 448

Appendix B. Elements of Universal Algebra 451

B.1. Abstract algebras 451
B.2. Boolean algebras 452
B.2.1. Boolean algebras of subsets 452
B.2.2. Boolean algebras 453
B.2.3. Stone's representation theorem 453
B.3. Binary relations and relation algebras 454
B.3.1. Binary relations 454
B.3.2. Inversion and composition 455
B.3.3. Proper relation algebras 456
B.3.4. Relation algebras 456
B.3.5. Representations 457
B.4. Basic elements of the language of categories 457
B.4.1. Categories and functors 457
B.4.2. Adjoint functors 459
B.4.3. Galois connections 461

Appendix C. Disjunctive Linear Relations 463

C.1. DLRs: definitions and satisfiability 463
C.2. Linear programming 464
C.3. Complexity of the satisfiability problem 466

Bibliography 471

Index 501