
Contents

Introduction	xi
Chapter 1. Integrals	1
1.1. Riemann integrals	3
1.2. Lebesgue integrals	6
1.3. Matlab® classes for a Riemann integral by trapezoidal integration	10
1.4. Matlab® classes for Lebesgue's integral	17
1.5. Matlab® classes for evaluation of the integrals when f is defined by a subprogram	33
1.6. Matlab® classes for partitions including the evaluation of the integrals	38
Chapter 2. Variational Methods for Algebraic Equations	51
2.1. Linear systems	52
2.2. Algebraic equations depending upon a parameter	62
2.2.1. Approximation of the solution by collocation	63
2.2.2. Variational approximation of the solution	65
2.2.3. Linear equations	66
2.2.4. Connection to orthogonal projections	67
2.2.5. Numerical determination of the orthogonal projections	69
2.2.6. Matlab® classes for a numerical solution	70
2.3. Exercises	98

Chapter 3. Hilbert Spaces for Engineers	103
3.1. Vector spaces	107
3.2. Distance, norm and scalar product	109
3.2.1. Distance	110
3.2.2. Norm	111
3.2.3. Scalar product	112
3.2.4. Cartesian products of vector spaces	117
3.2.5. A Matlab® class for scalar products and norms.	118
3.2.6. A Matlab® class for Gram–Schmidt orthonormalization	125
3.3. Continuous maps.	132
3.4. Sequences and convergence	134
3.4.1. Sequences	134
3.4.2. Convergence (or strong convergence).	134
3.4.3. Weak convergence	138
3.4.4. Compactness.	139
3.5. Hilbert spaces and completeness	141
3.5.1. Fixed points	142
3.6. Open and closed sets	144
3.6.1. Closure of a set	144
3.6.2. Open and closed sets	145
3.6.3. Dense subspaces	146
3.7. Orthogonal projection.	147
3.7.1. Orthogonal projection on a subspace	147
3.7.2. Orthogonal projection on a convex subset	150
3.7.3. Orthogonal projection on an affine subspace.	151
3.7.4. Matlab® determination of orthogonal projections.	153
3.8. Series and separable spaces	157
3.8.1. Series	158
3.8.2. Separable spaces and Fourier series	159
3.9. Duality.	161
3.9.1. Linear functionals.	161
3.9.2. Kernel of a linear functional	164
3.9.3. Riesz’s theorem	166
3.10. Generating a Hilbert basis	167
3.10.1. 1D situations	168
3.10.2. 2D situations	169
3.10.3. 3D situations	172
3.10.4. Using a sequence of finite families.	175
3.11. Exercises.	175

Chapter 4. Functional Spaces for Engineers	185
4.1. The $L^2(\Omega)$ space	186
4.2. Weak derivatives	189
4.2.1. Second-order weak derivatives	191
4.2.2. Gradient, divergence, Laplacian	192
4.2.3. Higher-order weak derivatives	195
4.2.4. Matlab® determination of weak derivatives	195
4.3. Sobolev spaces	199
4.3.1. Point values	201
4.4. Variational equations involving elements of a functional space	203
4.5. Reducing multiple indexes to a single one	205
4.6. Existence and uniqueness of the solution of a variational equation	207
4.7. Linear variational equations in separable spaces	210
4.8. Parametric variational equations	211
4.9. A Matlab® class for variational equations	213
4.10. Exercises	216
Chapter 5. Variational Methods for Differential Equations	221
5.1. A simple situation: the oscillator with one degree of freedom	224
5.1.1. Newton's equation of motion	225
5.1.2. Boundary value problem and initial condition problem	226
5.1.3. Generating a variational formulation	226
5.1.4. Generating an approximation of a variational equation	230
5.1.5. Application to the first variational formulation of the BVP	230
5.1.6. Application to the second variational formulation of the BVP	231
5.1.7. Application to the first variational formulation of the ICP	232
5.1.8. Application to the second variational formulation of the ICP	232
5.2. Connection between differential equations and variational equations	233

5.2.1. From a variational equation to a differential equation	233
5.2.2. From a differential equation to a variational equation	236
5.3. Variational approximation of differential equations	243
5.4. Evolution partial differential equations.	253
5.4.1. Linear equations.	253
5.4.2. Nonlinear equations	255
5.4.3. Motion equations	256
5.4.4. Motion of a bar	264
5.4.5. Motion of a beam under flexion	268
5.5. Exercises	272
Chapter 6. Dirac's Delta	279
6.1. A simple example	283
6.2. Functional definition of Dirac's delta	285
6.2.1. Compact support functions	285
6.2.2. Infinitely differentiable functions having a compact support	286
6.2.3. Formal definition of Dirac's delta	287
6.2.4. Dirac's delta as a probability	287
6.3. Approximations of Dirac's delta	288
6.4. Smoothed particle approximations of Dirac's delta	289
6.5. Derivation using Dirac's delta approximations	291
6.6. A Matlab® class for smoothed particle approximations	292
6.7. Green's functions	298
6.7.1. Adjoint operators	299
6.7.2. Green's functions	301
6.7.3. Using fundamental solutions to solve differential equations.	302
Chapter 7. Functionals and Calculus of Variations	319
7.1. Differentials.	320
7.2. Gâteaux derivatives of functionals	321
7.3. Convex functionals	324

7.4. Standard methods for the determination of Gâteaux derivatives	326
7.4.1. Methods for practical calculation	326
7.4.2. Gâteaux derivatives and equations of the motion of a system	329
7.4.3. Gâteaux derivatives of Lagrangians	332
7.4.4. Gâteaux derivatives of fields	333
7.5. Numerical evaluation and use of Gâteaux differentials	334
7.5.1. Numerical evaluation of a functional	335
7.5.2. Determination of a Gâteaux derivative	336
7.5.3. Determination of the derivatives with respect to the Hilbertian coefficients	339
7.5.4. Solving an equation involving the Gâteaux differential	343
7.5.5. Determining an optimal point	345
7.6. Minimum of the energy	347
7.7. Lagrange's multipliers	349
7.8. Primal and dual problems	352
7.9. Matlab® determination of minimum energy solutions	354
7.10. First-order control problems	366
7.11. Second-order control problems	371
7.12. A variational approach for multiobjective optimization	374
7.13. Matlab® implementation of the variational approach for biobjective optimization	384
7.14. Exercises	388
Bibliography	393
Index	411