

# Table of Contents

<b>Introduction</b> . . . . .	xiii
<b>PART 1. GENERAL CONSIDERATIONS CONCERNING NUMERICAL TOOLS</b> . . . . .	1
<b>Chapter 1. Feedback on the Notion of a Model and the Need for Calibration</b> . . . . .	3
Denis DARTUS	
1.1. “Static” and “dynamic” calibrations of a model . . . . .	6
1.1.1. Static calibration . . . . .	6
1.1.1.1. Static calibration methods . . . . .	6
1.1.1.2. Role of static calibration . . . . .	8
1.1.1.3. Problems associated with static calibration . . . . .	9
1.2. “Dynamic” calibration of a model or data assimilation . . . . .	10
1.3. Bibliography . . . . .	10
<b>Chapter 2. Engineering Model and Real-Time Model</b> . . . . .	11
Jean-Michel TANGUY	
2.1. Categories of modeling tools . . . . .	11
2.2. Weather forecasting at Météo France . . . . .	12
2.2.1. Objective analysis . . . . .	14
2.2.2. Expertise – publication (output of results) . . . . .	16
2.3. Flood forecasting . . . . .	18
2.4. Characteristics of real-time models . . . . .	23
2.5. Environment of real-time platforms . . . . .	25
2.6. Interpretation of hydrological forecasting by those responsible for civil protection . . . . .	27
2.7. Conclusion . . . . .	29
2.8. Bibliography . . . . .	30

<b>Chapter 3. From Mathematical Model to Numerical Model</b> . . . . .	31
Jean-Michel TANGUY	
3.1. Classification of the systems of differential equations . . . . .	32
3.2. 3D, 2D, 1D systems . . . . .	33
3.2.1. Reduction in the number of dimensions of the problem . . . . .	33
3.2.1.1. Two-dimensional horizontal model (2DH model) . . . . .	34
3.2.1.2. Two-dimensional vertical model (2DV model) . . . . .	36
3.2.1.3. One-dimensional (longitudinal) model (1D model) . . . . .	37
3.2.2. Removal of terms from the equations . . . . .	39
3.3. Discrete systems and continuous systems . . . . .	40
3.4. Equilibrium and propagation problems . . . . .	41
3.4.1. Permanent (equilibrium) or boundary value problems . . . . .	41
3.4.2. Propagation or transitory problems . . . . .	42
3.5. Linear and non-linear systems . . . . .	43
3.5.1. Systems of first- and second-order partial differential equations . . . . .	45
3.5.1.1. Introduction to the notion of characteristic . . . . .	45
3.5.2. Second-order hyperbolic, parabolic and elliptic equations . . . . .	46
3.5.2.1. Hyperbolic problems . . . . .	48
3.5.2.2. Parabolic problems . . . . .	50
3.5.2.3. Elliptic problems . . . . .	51
3.5.3. Applications of the characteristics method . . . . .	52
3.5.3.1. Additions complementing the method . . . . .	52
3.5.3.2. Super-critical and sub-critical flows with Saint-Venant's equation . . . . .	52
3.5.3.3. Numerical impacts with the non-linear convection equation . . . . .	55
3.5.3.4. Summary table of the equation types . . . . .	56
3.6. Conclusion . . . . .	57
3.7. Bibliography . . . . .	57
<b>PART 2. DISCRETIZATION METHODS</b> . . . . .	59
<b>Chapter 4. Problematic Issues Encountered</b> . . . . .	61
Marie-Madeleine MAUBOURGUET	
4.1. Examples of unstable problems . . . . .	62
4.1.1. Pure diffusion equation . . . . .	62
4.1.2. Saint-Venant 2DH equation . . . . .	63
4.2. Loss of material . . . . .	63
4.2.1. Navier-Stokes equations . . . . .	63
4.2.2. Saint-Venant 2DH equation . . . . .	65
4.3. Unsuitable scheme . . . . .	66
4.3.1. Diffusive scheme . . . . .	67
4.4. Bibliography . . . . .	69

<b>Chapter 5. General Presentation of Numerical Methods</b> . . . . .	71
Serge PIPERNO and Alexandre ERN	
5.1. Introduction . . . . .	71
5.2. Finite difference method . . . . .	72
5.2.1. Principles of the method . . . . .	72
5.2.2. Essential properties . . . . .	74
5.2.3. Extensions . . . . .	75
5.3. Finite volume method . . . . .	77
5.3.1. Introduction . . . . .	77
5.3.2. Principles of the method . . . . .	77
5.4. Finite element method . . . . .	78
5.4.1. Principles of the method . . . . .	79
5.4.2. Essential properties . . . . .	82
5.4.3. Evolution problems . . . . .	86
5.4.4. Discontinuous finite elements . . . . .	88
5.5. Comparison of the different methods on a convection/diffusion problem . . . . .	92
5.6. Bibliography . . . . .	93
<b>Chapter 6. Finite Differences</b> . . . . .	95
Marie-Madeleine MAUBOURGUET and Jean-Michel TANGUY	
6.1. General principles of the finite difference method . . . . .	95
6.2. Discretization of initial and boundary conditions . . . . .	102
6.2.1. Neumann condition . . . . .	103
6.3. Resolution on a 2D domain . . . . .	105
6.3.1. Summary . . . . .	107
<b>Chapter 7. Introduction to the Finite Element Method</b> . . . . .	109
Jean-Michel TANGUY	
7.1. Elementary FEM concepts and presentation of the section . . . . .	109
7.2. Method of approximation by finite elements . . . . .	111
7.2.1. Definitions . . . . .	112
7.2.2. Rule for partitioning the domain into elements . . . . .	114
7.3. Geometric transformation . . . . .	114
7.3.1. Notion of a reference element in one dimension . . . . .	114
7.3.2. Expression using overall coordinates . . . . .	115
7.3.3. Expression using local coordinates of the element . . . . .	115
7.3.4. Expression using local “reference” coordinates . . . . .	115
7.3.5. 2D approach on a three-node triangular element . . . . .	118
7.3.6. General approach . . . . .	119
7.4. Transformation of derivation and integration operators . . . . .	121
7.4.1. Transformation of derivation operators . . . . .	121
7.4.2. Expression of the Jacobian matrix $[J]$ and its inverse $[j]$ . . . . .	123

7.4.3. Transformation of an integral . . . . .	125
7.5. Geometric definition of the elements . . . . .	125
7.6. Method of weighted residuals . . . . .	128
7.7. Transformation of integral forms . . . . .	130
7.7.1. Integration by parts . . . . .	130
7.7.2. Weak integral form . . . . .	131
7.8. Matrix presentation of the finite element method . . . . .	133
7.8.1. Finite element method . . . . .	133
7.8.2. Discretized elementary integral forms of $W^e$ . . . . .	137
7.8.2.1. Matrix expression of $W^e$ . . . . .	137
7.8.2.2. Case of a non-linear operator $L$ . . . . .	140
7.9. Integral form of $W^e$ on the reference element . . . . .	140
7.9.1. Transformation of derivations . . . . .	140
7.9.2. Transformation of the integration domain . . . . .	141
7.9.3. A few conventional forms of $W^e$ and elementary matrices . . . . .	141
7.9.4. Assembly of the discretized overall form $W$ . . . . .	144
7.9.4.1. Overall and elementary variables . . . . .	145
7.9.4.2. Elementary $\{u_n\}$ and overall $\{U_n\}$ vectors . . . . .	145
7.10. Introduction of the Dirichlet-type boundary conditions . . . . .	148
7.10.1. Dominant diagonal term method . . . . .	148
7.10.2. Unit term on the diagonal method . . . . .	149
7.10.3. Equation removal method . . . . .	150
7.11. Summary: implementation of the finite element method . . . . .	151
7.12. Application example: wave propagation . . . . .	151
7.12.1. Berkhoff equations . . . . .	152
7.12.2. Boundary conditions . . . . .	153
7.12.3. Integral formulation . . . . .	155
7.13. Bibliography . . . . .	158
<b>Chapter 8. Presentation of the Finite Volume Method . . . . .</b>	<b>161</b>
Alexandre ERN and Serge PIPERNO, section 8.6 written by Dominique THIÉRY	
8.1. 1D conservation equations . . . . .	162
8.1.1. 1D scalar conservation laws . . . . .	163
8.1.2. Systems of 1D conservation laws . . . . .	167
8.2. Classical, weak and entropic solutions . . . . .	170
8.2.1. Introduction . . . . .	170
8.2.2. Weak solutions of the conservation equation . . . . .	170
8.2.3. Entropy conditions, entropic solutions . . . . .	172
8.3. Numerical solution of a conservation law . . . . .	175
8.3.1. Finite volume method . . . . .	175
8.3.2. Godunov method . . . . .	177
8.3.3. Examples of Godunov methods . . . . .	180

8.3.4. Complete solution algorithm for the traffic model . . . . .	181
8.3.5. Approximate Riemann solvers . . . . .	182
8.4. Numerical solution of hyperbolic systems . . . . .	183
8.4.1. 1D cases . . . . .	183
8.4.2. Approximate Riemann solvers . . . . .	187
8.4.3. 2D finite volume method . . . . .	189
8.4.4. Complete solution algorithm for a two-dimensional problem . . . . .	192
8.5. High-order, finite volume methods . . . . .	194
8.6. Application of the finite volume method to the flow development of groundwater . . . . .	195
8.6.1. Confined aquifer with a meshing formed by uniform cubes . . . . .	196
8.6.1.1. Homogeneous aquifer system, no source term, under permanent flow . . . . .	199
8.6.1.2. Aquifer system with no source term, under steady state flow . . . . .	199
8.6.1.3. Aquifer system with no source term . . . . .	200
8.6.1.4. General case with a source term . . . . .	201
8.6.2. Confined aquifer with a meshing formed by irregular parallelepipeds . . . . .	201
8.6.3. Monolayer unconfined aquifer with a meshing formed by irregular parallelepipeds . . . . .	202
8.6.4. Systems of equations and resolution . . . . .	203
8.6.5. Resolution of non-linear systems . . . . .	204
8.6.6. Computing exchange coefficients between two adjacent meshes . . . . .	204
8.6.7. Taking the boundary conditions into account . . . . .	207
8.6.7.1. Processing an impervious limit . . . . .	207
8.6.7.2. Processing a prescribed head mesh . . . . .	207
8.6.7.3. Introducing an exchange flow onto a limit . . . . .	207
8.6.8. Extending the finite volume method to more complex meshing . . . . .	207
8.6.8.1. Columns and rows with variable dimensions . . . . .	208
8.6.8.2. Meshes that are no longer parallelepipeds or hexahedrons . . . . .	208
8.6.8.3. Nested meshings . . . . .	208
8.7. Bibliography . . . . .	210
<b>Chapter 9. Spectral Methods in Meteorology . . . . .</b>	<b>213</b>
Jean COIFFIER	
9.1. Introduction . . . . .	213
9.2. Using finite series expansion of functions . . . . .	214
9.2.1. General ideas about Galerkin methods . . . . .	214
9.2.2. The various applications of the Galerkin method . . . . .	215

9.3. The spectral method on the sphere . . . . .	216
9.3.1. Historical background . . . . .	216
9.3.2. The basis of surface spherical harmonics . . . . .	216
9.3.3. Properties of the spherical harmonics . . . . .	218
9.3.4. Expansion of a spherical field . . . . .	220
9.3.5. Truncated expansion . . . . .	221
9.3.6. Computing linear terms . . . . .	222
9.3.7. Computing non-linear terms . . . . .	223
9.3.8. Practical implementation of the spectral method . . . . .	226
9.4. The spectral method on a bi-periodic domain . . . . .	227
9.4.1. Constructing a bi-periodic domain . . . . .	227
9.4.2. The basis functions . . . . .	228
9.4.3. Elliptic truncation . . . . .	230
9.4.4. Computing linear terms . . . . .	231
9.4.5. Computing non-linear terms . . . . .	231
9.4.6. Benefits of the method . . . . .	232
9.5. Bibliography . . . . .	232
<b>Chapter 10. Numerical-Scheme Study . . . . .</b>	<b>235</b>
Jean-Michel TANGUY	
10.1. Reminder of the notion of the numerical scheme . . . . .	235
10.2. Time discretization . . . . .	236
10.2.1. First-order temporal discretization: semi-implicit scheme . . . . .	236
10.2.2. Second-order temporal discretization: explicit scheme . . . . .	237
10.2.3. Third-order temporal discretization: explicit scheme . . . . .	238
10.2.4. First-order temporal discretization: implicit scheme . . . . .	240
10.3. Space discretization . . . . .	240
10.4. Scheme study: notions of consistency, stability and convergence . . . . .	241
10.4.1. Truncation error – consistency . . . . .	242
10.4.2. Stability . . . . .	243
10.4.2.1. Von Neumann method . . . . .	244
10.4.2.2. Applications on the one-dimensional convection equation . . . . .	245
10.4.2.3. Comment regarding the CFL (Courant, Friedrich and Levy) condition . . . . .	250
10.4.2.4. Modified-equation method . . . . .	251
10.4.2.5. Stability study based on the modified equation . . . . .	252
10.4.2.6. Explicit Euler scheme centered with respect to space . . . . .	253
10.4.2.7. Lax-Wendroff scheme (2nd order) . . . . .	255
10.4.2.8. Behavior of the 2nd-order LW scheme (LW2), applied to the 1D convection equation . . . . .	255
10.4.2.9. Implicit Euler scheme . . . . .	256
10.4.2.10. Matrix method . . . . .	258

10.4.3. Convergence . . . . .	260
10.4.4. Example: study of a numerical scheme applied to a PDE . . . . .	262
10.4.4.1. Summary table of the properties of the schemes studied . . . . .	262
10.5. Bibliography . . . . .	264
<b>Chapter 11. Resolution Methods . . . . .</b>	<b>267</b>
Marie-Madeleine MAUBOURGUET	
11.1. Temporal integration methods . . . . .	268
11.2. Linearization methods for non-linear systems . . . . .	270
11.3. Methods for solving linear systems $AX = B$ . . . . .	271
11.3.1. Direct methods . . . . .	271
11.3.2. Iterative methods . . . . .	271
11.4. Bibliography . . . . .	272
<b>PART 3. INTRODUCTION TO DATA ASSIMILATION . . . . .</b>	<b>273</b>
<b>Chapter 12. Data Assimilation . . . . .</b>	<b>275</b>
Jean PAILLEUX, Denis DARTUS, Xijun LAI, Jérôme MONNIER and Marc HONNORAT	
12.1. Several examples of the application of data assimilation . . . . .	277
12.1.1. Data assimilation in meteorology . . . . .	277
12.1.2. Data assimilation in hydrology . . . . .	280
12.1.2.1. Global sensitivity analysis . . . . .	281
12.1.2.2. Temporal sensitivity analysis . . . . .	281
12.1.2.3. Spatial sensitivity analysis . . . . .	282
12.1.2.4. Identification of the Richards parameters . . . . .	282
12.2. Data assimilation in hydraulics with the Dassflow model . . . . .	284
12.2.1. Example of the Pearl River . . . . .	287
12.3. Bibliography . . . . .	290
<b>Chapter 13. Data Assimilation Methodology . . . . .</b>	<b>295</b>
Hélène BESSIÈRE, Hélène ROUX, François-Xavier LE DIMET and Denis DARTUS	
13.1. Representation of the system . . . . .	295
13.2. Taking errors into account . . . . .	296
13.3. Simplified approach to optimum static estimation theory . . . . .	297
13.3.1. First approach: minimization of the variance in the estimation error . . . . .	298
13.3.2. Second approach: weighted least squares . . . . .	299
13.4. Generalization in the multidimensional case . . . . .	300
13.4.1. Minimization of the variance of the linear estimator with background . . . . .	301
13.4.2. Weighted least squares . . . . .	302

13.5. The different data assimilation techniques . . . . .	303
13.6. Sequential assimilation method: the Kalman filter . . . . .	304
13.7. Extension to non-linear models: the extended Kalman filter . . . . .	307
13.8. Assessment of the Kalman filter . . . . .	308
13.9. Variational methods . . . . .	312
13.10. Discreet formulation of the cost function: the 3D-VAR . . . . .	313
13.11. General variational formalism: the 4D-VAR . . . . .	314
13.12. Continuous formulation of the cost function . . . . .	314
13.12.1. The adjoint method . . . . .	316
13.13. Principle of automatic differentiation . . . . .	322
13.14. Summary of variational methods . . . . .	322
13.15. A complete application example: the Burgers equation . . . . .	324
13.15.1. Analytical resolution using the adjoint method . . . . .	325
13.15.2. Using automatic differentiation . . . . .	331
13.16. Feedback on the notion of a model and the need for calibration . . . . .	335
13.16.1. Modeling guidelines, adapted from Schlesinger . . . . .	336
13.16.2. Static calibration of a model . . . . .	339
13.16.2.1. Static calibration methods . . . . .	339
13.16.2.2. Role of static calibration . . . . .	341
13.16.2.3. Problems associated with static calibration . . . . .	342
13.16.3. “Dynamic” calibration of a model or data assimilation . . . . .	343
13.17. Bibliography . . . . .	343
<b>List of Authors . . . . .</b>	<b>349</b>
<b>Index . . . . .</b>	<b>351</b>
<b>General Index of Authors . . . . .</b>	<b>353</b>
<b>Summary of the Other Volumes in the Series . . . . .</b>	<b>355</b>