Contents

Preface xi

Errata for Volume 1 and Volume 2 xv

List of Notations xix

Chapter 1. Differential Calculus 1

1.1. Introduction 1
1.2. Fréchet differential calculus 2
1.2.1. General conventions 2
1.2.2. Fréchet differential 5
1.2.3. Mappings of class C^p 9
1.2.4. Taylor's formulas 12
1.2.5. Analytic functions 16
1.2.6. The implicit function theorem and its consequences 19
1.3. Other approaches to differential calculus 27
1.3.1. Lagrange variations and Gateaux differentials 27
1.3.2. Calculus of variations: elementary concepts 29
1.3.3. "Convenient" differentials 32
1.4. Smooth partitions of unity 35
1.4.1. C^∞-paracompactness of Banach spaces 35
1.4.2. $\mathfrak{c}^\infty$-paracompactness 36
1.5. Ordinary differential equations 37
1.5.1. Existence and uniqueness theorems 37
1.5.2. Linear differential equations 43
1.5.3. Parameter dependence of solutions 45

Chapter 2. Differential and Analytic Manifolds 49

2.1. Introduction 49
2.2. Manifolds: tangent space of a manifold at a point 50
2.2.1. Notion of a manifold 50
2.2.2. Morphisms of manifolds 56
2.2.3. Tangent mappings 58
2.2.4. Tangent vectors 58
2.3. Tangent linear mappings; submanifolds 65
2.3.1. Tangent linear mapping; rank 65
2.3.2. Differential 66
2.3.3. Submanifolds 67
2.3.4. Immersions and embeddings 68
2.3.5. Submersions, subimmersions and *étale* mappings 71
2.3.6. Submanifolds of $\mathbb{K}^n$ 74
2.3.7. Products of manifolds 75
2.3.8. Transversal morphisms and manifolds 76
2.3.9. Fiber product of manifolds 78
2.3.10. Covectors and cotangent spaces 79
2.3.11. Cotangent linear mapping 80
2.4. Lie groups and their actions 81
2.4.1. Lie groups 81
2.4.2. Manifolds of orbits and homogeneous manifolds 88

Chapter 3. Fiber Bundles 93

3.1. Introduction 93
3.2. Tangent bundle and cotangent bundle 94
3.2.1. Tangent bundle 94
3.2.2. Cotangent bundle 96
3.2.3. Tangent bundle and cotangent bundle functors 98
3.3. Fibrations 98
3.3.1. Notion of a fibration 99
3.3.2. Fiber product and preimage of fibrations 101
3.3.3. Coverings 103
3.3.4. Sections 107
3.4. Vector bundles 108
3.4.1. Vector bundles 108
3.4.2. Dual of a vector bundle 112
3.4.3. Subbundles and quotient bundles 113
3.4.4. Whitney sum and tensor product 114
3.4.5. The category of vector bundles 115
3.4.6. Preimage of a fiber bundle 120
3.5. Principal bundles 121
3.5.1. Notion of a principal bundle 121

3.5.2. Vertical tangent vectors . . . 123
3.5.3. Morphisms of principal bundles . . . 124
3.5.4. Principal bundles defined by cocycles . . . 124
3.5.5. Fiber bundle associated with a principal bundle . . . 125
3.5.6. Extension, restriction, quotientization of the structural group . . . 126
3.5.7. Examples of trivial principal bundles . . . 128

Chapter 4. Tensor Calculus on Manifolds . . . 131

4.1. Introduction . . . 131
4.2. Tensor calculus . . . 132
4.2.1. Tensors . . . 132
4.2.2. Symmetric tensors and antisymmetric tensors . . . 135
4.2.3. Exterior algebra . . . 138
4.2.4. Duality in the exterior algebra . . . 139
4.2.5. Interior products . . . 141
4.2.6. Tensors on Banach spaces . . . 143
4.3. Tensor fields . . . 145
4.3.1. Vector fields . . . 145
4.3.2. Covector field . . . 146
4.3.3. Tensor fields and scalar fields . . . 146
4.4. Differential forms . . . 148
4.4.1. Differential forms of degree p . . . 148
4.4.2. Preimage of a differential p-form . . . 149
4.4.3. Differential forms taking values in a fiber bundle. List of formulas . . . 151
4.4.4. Orientation . . . 154
4.4.5. Integral of a differential form of maximal degree . . . 157
4.4.6. Differential forms of odd type . . . 163
4.4.7. Integration of a differential form over a chain . . . 166
4.5. Pseudo-Riemannian manifolds . . . 170
4.5.1. Metric . . . 170
4.5.2. Pseudo-Riemannian volume element . . . 171

Chapter 5. Differential and Integral Calculus on Manifolds . . . 173

5.1. Introduction . . . 173
5.2. Currents and differential operators . . . 174
5.2.1. Currents and distributions . . . 174
5.2.2. Differential operators and point distributions . . . 181
5.3. Manifolds of mappings . . . 183
5.3.1. The Banach framework . . . 183
5.3.2. The “convenient” framework . . . 186
5.4. Lie derivatives . . . 187

5.4.1. Lie algebras . . . 187
5.4.2. Lie derivative of a function . . . 190
5.4.3. Lie brackets . . . 192
5.4.4. Lie derivative of vector, covector and tensor fields . . . 193
5.4.5. Lie derivative of a p-form . . . 194
5.5. Exterior differential . . . 195
5.5.1. É. Cartan's theorem . . . 195
5.5.2. Application to vector calculus . . . 198
5.6. Stokes' formula and applications . . . 200
5.6.1. Stokes' formula on a chain . . . 200
5.6.2. Ostrogradsky and Green formulas . . . 203
5.6.3. Hodge duality and codifferentials . . . 206
5.6.4. Gauss' theorem and Poisson's formula . . . 213
5.6.5. Homology, cohomology and duality . . . 215
5.7. Integral curves and manifolds . . . 224
5.7.1. First-order differential equations . . . 224
5.7.2. Second-order differential equations . . . 228
5.7.3. Sprays . . . 229
5.7.4. Straightening of vector fields and frames . . . 231
5.7.5. Integral manifolds, foliations . . . 233

Chapter 6. Analysis on Lie Groups . . . 245

6.1. Introduction . . . 245
6.2. Convolution . . . 246
6.2.1. Convolution of distributions . . . 246
6.2.2. Haar measure and convolution of functions . . . 250
6.3. Classification of Lie algebras . . . 256
6.3.1. Additional notions from algebra . . . 256
6.3.2. Classical Lie algebras . . . 259
6.3.3. General notions about Lie algebras . . . 260
6.3.4. Nilpotent Lie algebras . . . 263
6.3.5. Solvable Lie algebras . . . 265
6.3.6. Simple and semi-simple Lie algebras . . . 267
6.3.7. Reductive Lie algebras . . . 271
6.3.8. Real compact Lie algebras . . . 272
6.4. Relation between Lie groups and Lie algebras . . . 273
6.4.1. Lie algebra of a Lie group . . . 273
6.4.2. Passing from a Lie algebra to a Lie group . . . 278
6.4.3. Dictionary . . . 281
6.5. Harmonic analysis . . . 284
6.5.1. Introduction . . . 284
6.5.2. Harmonic analysis on $\mathbb{R}^n$ . . . 286
6.5.3. Fourier series and Fourier transforms on the torus . . . 296

6.5.4. Fourier transform on a locally compact commutative group . . . 302
6.5.5. Overview of non-commutative harmonic analysis . . . 310

Chapter 7. Connections . . . 315

7.1. Introduction . . . 315
7.2. Linear connections . . . 317
7.2.1. Curvilinear coordinates . . . 317
7.2.2. Linear connection on a vector bundle . . . 323
7.2.3. Linear connection on a manifold . . . 325
7.2.4. Parallel transport and geodesics . . . 327
7.2.5. Covariant exterior differential . . . 330
7.2.6. Curvature and torsion of a linear connection . . . 331
7.3. Method of moving frames . . . 333
7.3.1. Moving frame and gauge potential . . . 334
7.3.2. Curvature, torsion and covariant exterior differential of a $\mathbf{G}$-connection . . . 337
7.3.3. Quasi-parallelogram method . . . 340
7.3.4. Fundamental equalities . . . 344
7.3.5. Connection form on the bundle of $\mathbf{G}$-frames . . . 345
7.3.6. Principal connections and parallel transport . . . 347
7.3.7. Covariant exterior differentiation on a principal bundle . . . 350
7.3.8. Characterization of a $\mathbf{G}$-connection . . . 351
7.3.9. Curvature and torsion forms of a principal connection . . . 352
7.3.10. Cartan connections . . . 355
7.4. Riemannian geometry . . . 358
7.4.1. Levi-Cività connection . . . 358
7.4.2. Geodesics . . . 360
7.4.3. Flat pseudo-Riemannian manifolds . . . 361
7.4.4. Ricci tensor and Einstein tensor . . . 363

References . . . 369

Cited Authors . . . 379

Index . . . 387