Contents

Foreword xi
Abdenacer MAKHLOUF

Chapter 1. Jordan Superalgebras 1
Consuelo MARTINEZ and Efim ZELMANOV

1.1. Introduction 1
1.2. Tits–Kantor–Koecher construction 4
1.3. Basic examples (classical superalgebras) 5
1.4. Brackets 8
1.5. Cheng–Kac superalgebras 10
1.6. Finite dimensional simple Jordan superalgebras 11
1.6.1. Case F is algebraically closed and $\mathrm{char}\, F = 0$ 11
1.6.2. Case $\mathrm{char}\, F = p > 2$, the even part $J_{\bar{0}}$ is semisimple 11
1.6.3. Case $\mathrm{char}\, F = p > 2$, the even part $J_{\bar{0}}$ is not semisimple 13
1.6.4. Non-unital simple Jordan superalgebras 13
1.7. Finite dimensional representations 14
1.7.1. Superalgebras of rank ≥ 3 16
1.7.2. Superalgebras of rank ≤ 2 17
1.8. Jordan superconformal algebras 21
1.9. References 23

Chapter 2. Composition Algebras 27
Alberto ELDUQUE

2.1. Introduction 27
2.2. Quaternions and octonions 28
2.2.1. Quaternions 28

2.2.2. Rotations in three- (and four-) dimensional space 31
2.2.3. Octonions 33
2.3. Unital composition algebras 35
2.3.1. The Cayley-Dickson doubling process and the generalized Hurwitz theorem 37
2.3.2. Isotropic Hurwitz algebras 41
2.4. Symmetric composition algebras 43
2.5. Triality 50
2.6. Concluding remarks 54
2.7. Acknowledgments 55
2.8. References 55

Chapter 3. Graded-Division Algebras 59
Yuri BAHTURIN, Mikhail KOCHETOV and Mikhail ZAICEV

3.1. Introduction 59
3.2. Background on gradings 62
3.2.1. Gradings induced by a group homomorphism 62
3.2.2. Weak isomorphism and equivalence 63
3.2.3. Basic properties of division gradings 63
3.2.4. Graded presentations of associative algebras 64
3.2.5. Tensor products of division gradings 68
3.2.6. Loop construction 70
3.2.7. Another construction of graded-simple algebras 72
3.3. Graded-division algebras over algebraically closed fields 75
3.4. Real graded-division associative algebras 77
3.4.1. Simple graded-division algebras 77
3.4.2. Pauli gradings 80
3.4.3. Commutative case 80
3.4.4. Non-commutative graded-division algebras with one-dimensional homogeneous components 82
3.4.5. Equivalence classes of graded-division algebras with one-dimensional homogeneous components 84
3.4.6. Graded-division algebras with non-central two-dimensional identity components 90
3.4.7. Graded-division algebras with four-dimensional identity components 94
3.4.8. Classification of real graded-division algebras, up to isomorphism 95
3.5. Real loop algebras with a non-split centroid 96
3.6. Alternative algebras 98
3.6.1. Cayley–Dickson doubling process 99
3.6.2. Gradings on octonion algebras 100
3.6.3. Graded-simple real alternative algebras 101

3.6.4. Graded-division real alternative algebras 102
3.7. Gradings of fields 106
3.8. References 107

Chapter 4. Non-associative C^*-algebras 111
Ángel RODRÍGUEZ PALACIOS and Miguel CABRERA GARCÍA

4.1. Introduction 111
4.2. *JB*-algebras 111
4.3. The non-associative Vidav–Palmer and Gelfand–Naimark theorems . . 116
4.4. JB^*-triples 128
4.5. Past, present and future of non-associative C^*-algebras 141
4.6. Acknowledgments 145
4.7. References 145

Chapter 5. Structure of $H^\star$-algebras 155
José Antonio CUENCA MIRA

5.1. Introduction 155
5.2. Preliminaries: aspects of the general theory 156
5.3. Ultraproducts of $H^\star$-algebras 164
5.4. Quadratic $H^\star$-algebras 166
5.5. Associative $H^\star$-algebras 167
5.6. Flexible $H^\star$-algebras 173
5.7. Non-commutative Jordan $H^\star$-algebras 175
5.8. Jordan $H^\star$-algebras 178
5.9. Moufang $H^\star$-algebras 182
5.10. Lie $H^\star$-algebras 184
5.11. Topics closely related to Lie $H^\star$-algebras 188
5.12. Two-graded $H^\star$-algebras 190
5.13. Other topics: beyond the $H^\star$-algebras 194
5.14. Acknowledgments 194
5.15. References 194

Chapter 6. Krichever–Novikov Type Algebras: Definitions and Results 199
Martin SCHLICHENMAIER

6.1. Introduction 199
6.2. The Virasoro algebra and its relatives 201
6.3. The geometric picture 204
6.3.1. The geometric realizations of the Witt algebra 204
6.3.2. Arbitrary genus generalizations 204
6.3.3. Meromorphic forms 206
6.4. Algebraic structures 209

6.4.1. Associative structure 209
6.4.2. Lie and Poisson algebra structure 210
6.4.3. The vector field algebra and the Lie derivative 210
6.4.4. The algebra of differential operators 211
6.4.5. Differential operators of all degrees 212
6.4.6. Lie superalgebras of half forms 213
6.4.7. Jordan superalgebra 213
6.4.8. Higher genus current algebras 214
6.4.9. KN-type algebras 215
6.5. Almost-graded structure 215
6.5.1. Definition of almost-gradedness 215
6.5.2. Separating cycle and KN pairing 216
6.5.3. The homogeneous subspaces 217
6.5.4. The algebras 219
6.5.5. Triangular decomposition and filtrations 221
6.6. Central extensions 221
6.6.1. Central extensions and cocycles 222
6.6.2. Geometric cocycles 223
6.6.3. Uniqueness and classification of central extensions 226
6.7. Examples and generalizations 229
6.7.1. The genus zero and three-point situation 229
6.7.2. Genus zero multipoint algebras – integrable systems 231
6.7.3. Deformations 232
6.8. Lax operator algebras 232
6.9. Fermionic Fock space 235
6.9.1. Semi-infinite forms and fermionic Fock space representations . . . 235
6.9.2. $b-c$ systems 237
6.10. Sugawara representation 237
6.11. Application to moduli space 240
6.12. Acknowledgments 240
6.13. References 240

Chapter 7. An Introduction to Pre-Lie Algebras 245
Chengming BAI

7.1. Introduction 245
7.1.1. Explanation of notions 245
7.1.2. Two fundamental properties 246
7.1.3. Some subclasses 247
7.1.4. Organization of this chapter 248
7.2. Some appearances of pre-Lie algebras 249
7.2.1. Left-invariant affine structures on Lie groups: a geometric interpretation of “left-symmetry” 249

7.2.2. Deformation complexes of algebras and right-symmetric algebras . . . 250
7.2.3. Rooted tree algebras: free pre-Lie algebras . . . 251
7.2.4. Complex structures on Lie algebras . . . 251
7.2.5. Symplectic structures on Lie groups and Lie algebras, phase spaces of Lie algebras and Kähler structures . . . 252
7.2.6. Vertex algebras . . . 254
7.3. Some basic results and constructions of pre-Lie algebras . . . 255
7.3.1. Some basic results of pre-Lie algebras . . . 255
7.3.2. Constructions of pre-Lie algebras from some known structures . . . 258
7.4. Pre-Lie algebras and CYBE . . . 261
7.4.1. The existence of a compatible pre-Lie algebra on a Lie algebra . . . 261
7.4.2. CYBE: unification of tensor and operator forms . . . 262
7.4.3. Pre-Lie algebras, $\mathcal{O}$-operators and CYBE . . . 264
7.4.4. An algebraic interpretation of "left-symmetry": construction from Lie algebras revisited . . . 265
7.5. A larger framework: Lie analogues of Loday algebras . . . 266
7.5.1. Pre-Lie algebras, dendriform algebras and Loday algebras . . . 266
7.5.2. L-dendriform algebras . . . 267
7.5.3. Lie analogues of Loday algebras . . . 269
7.6. References . . . 271

Chapter 8. Symplectic, Product and Complex Structures on 3-Lie Algebras . . . 275
Yunhe SHENG and Rong TANG

8.1. Introduction . . . 275
8.2. Preliminaries . . . 278
8.3. Representations of 3-pre-Lie algebras . . . 280
8.4. Symplectic structures and phase spaces of 3-Lie algebras . . . 282
8.5. Product structures on 3-Lie algebras . . . 288
8.6. Complex structures on 3-Lie algebras . . . 295
8.7. Complex product structures on 3-Lie algebras . . . 304
8.8. Para-Kähler structures on 3-Lie algebras . . . 308
8.9. Pseudo-Kähler structures on 3-Lie algebras . . . 315
8.10. References . . . 317

Chapter 9. Derived Categories . . . 321
Bernhard KELLER

9.1. Introduction . . . 321
9.2. Grothendieck's definition . . . 322
9.3. Verdier's definition . . . 323
9.4. Triangulated structure . . . 326
9.5. Derived functors . . . 331

9.6. Derived Morita theory 332
9.7. Dg categories 334
9.7.1. Dg categories and functors 334
9.7.2. The derived category 336
9.7.3. Derived functors 337
9.7.4. Dg quotients 338
9.7.5. Invariants 340
9.8. References 342

List of Authors 347

Index 349