
Contents

Introduction	xi
Familiarization with Semi-normed Spaces	xv
Notations	xvii
Chapter 1. Prerequisites	1
1.1. Sets, mappings, orders	1
1.2. Countability	3
1.3. Construction of \mathbb{R}	4
1.4. Properties of \mathbb{R}	5
Part 1. Semi-normed Spaces	9
Chapter 2. Semi-normed Spaces	11
2.1. Definition of semi-normed spaces	11
2.2. Convergent sequences	15
2.3. Bounded, open and closed sets	17
2.4. Interior, closure, balls and semi-balls	21
2.5. Density, separability	23
2.6. Compact sets	25
2.7. Connected and convex sets	30
Chapter 3. Comparison of Semi-normed Spaces	33
3.1. Equivalent families of semi-norms	33
3.2. Topological equalities and inclusions	34
3.3. Topological subspaces	39

3.4. Filtering families of semi-norms	43
3.5. Sums of sets	46
Chapter 4. Banach, Fréchet and Neumann Spaces	49
4.1. Metrizable spaces	49
4.2. Properties of sets in metrizable spaces	51
4.3. Banach, Fréchet and Neumann spaces	55
4.4. Compacts sets in Fréchet spaces	57
4.5. Properties of \mathbb{R}	58
4.6. Convergent sequences	60
4.7. Sequential completion of a semi-normed space	62
Chapter 5. Hilbert Spaces	65
5.1. Hilbert spaces	65
5.2. Projection in a Hilbert space	68
5.3. The space \mathbb{R}^d	70
Chapter 6. Product, Intersection, Sum and Quotient of Spaces	73
6.1. Product of semi-normed spaces	73
6.2. Product of a semi-normed space by itself	78
6.3. Intersection of semi-normed spaces	80
6.4. Sum of semi-normed spaces	83
6.5. Direct sum of semi-normed spaces	89
6.6. Quotient space	93
Part 2. Continuous Mappings	95
Chapter 7. Continuous Mappings	97
7.1. Continuous mappings	97
7.2. Continuity and change of topology or restriction	100
7.3. Continuity of composite mappings	102
7.4. Continuous semi-norms	102
7.5. Continuous linear mappings	104
7.6. Continuous multilinear mappings	107
7.7. Some continuous mappings	111
Chapter 8. Images of Sets Under Continuous Mappings	115
8.1. Images of open and closed sets	115
8.2. Images of dense, separable and connected sets	117
8.3. Images of compact sets	119

8.4. Images under continuous linear mappings	121
8.5. Continuous mappings in compact sets	123
8.6. Continuous real mappings	124
8.7. Compacting mappings	125
Chapter 9. Properties of Mappings in Metrizable Spaces	129
9.1. Continuous mappings in metrizable spaces	129
9.2. Banach's fixed point theorem	133
9.3. Baire's theorem	134
9.4. Open mapping theorem	136
9.5. Banach–Schauder's continuity theorem	138
9.6. Closed graph theorem	139
Chapter 10. Extension of Mappings, Equicontinuity	141
10.1. Extension of equalities by continuity	141
10.2. Continuous extension of mappings	142
10.3. Equicontinuous families of mappings	146
10.4. Banach–Steinhaus equicontinuity theorem	148
Chapter 11. Compactness in Mapping Spaces	153
11.1. The spaces $\mathcal{F}(X; F)$ and $\mathcal{C}(X; F)$ -pt	153
11.2. Zorn's lemma	154
11.3. Compactness in $\mathcal{F}(X; F)$	157
11.4. An Ascoli compactness theorem in $\mathcal{C}(X; F)$ -pt	161
Chapter 12. Spaces of Linear or Multilinear Mappings	163
12.1. The space $\mathcal{L}(E; F)$	163
12.2. Bounded sets in $\mathcal{L}(E; F)$	165
12.3. Sequential completeness of $\mathcal{L}(E; F)$ when E is metrizable	167
12.4. Semi-norms and norm on $\mathcal{L}(E; F)$ when E is normed	169
12.5. Continuity of the composition of linear mappings	171
12.6. Inversibility in the neighborhood of an isomorphism	174
12.7. The space $\mathcal{L}^d(E_1 \times \cdots \times E_d; F)$	178
12.8. Separation of the variables of a multilinear mapping	181
Part 3. Weak Topologies	187
Chapter 13. Duality	189
13.1. Dual	189
13.2. Dual of a metrizable or normed space	193
13.3. Dual of a Hilbert space	196

13.4. Extraction of * weakly converging subsequences	199
13.5. Continuity of the bilinear form of duality	203
13.6. Dual of a product	205
13.7. Dual of a direct sum	206
Chapter 14. Dual of a Subspace	209
14.1. Hahn–Banach theorem	209
14.2. Corollaries of the Hahn–Banach theorem	211
14.3. Characterization of a dense subspace	212
14.4. Dual of a subspace	213
14.5. Dual of an intersection	215
14.6. Dangerous identifications	216
Chapter 15. Weak Topology	221
15.1. Weak topology	221
15.2. Weak continuity and topological inclusions	224
15.3. Weak topology of a product	225
15.4. Weak topology of an intersection	226
15.5. Norm and semi-norms of a weak limit	228
Chapter 16. Properties of Sets for the Weak Topology	231
16.1. Banach–Mackey theorem (weakly bounded sets)	231
16.2. Gauge of a convex open set	233
16.3. Mazur’s theorem (weakly closed convex sets)	235
16.4. Šmulian’s theorem (weakly compact sets)	237
16.5. Semi-weak continuity of a bilinear mapping	240
Chapter 17. Reflexivity	243
17.1. Reflexive spaces	243
17.2. Sequential completion of a semi-reflexive space	247
17.3. Prereflexivity of metrizable spaces	248
17.4. Reflexivity of Hilbert spaces	250
17.5. Reflexivity of uniformly convex Banach spaces	252
17.6. A property of the combinations of linear forms	256
17.7. Characterizations of semi-reflexivity	257
17.8. Reflexivity of a subspace	261
17.9. Reflexivity of the image of a space	261
17.10. Reflexivity of the dual	263
Chapter 18. Extractable Spaces	265
18.1. Extractable spaces	265
18.2. Extractability of Hilbert spaces	266

18.3. Extractability of semi-reflexive spaces	267
18.4. Extractability of a subspace or of the image of a space	269
18.5. Extractability of a product or of a sum of spaces	270
18.6. Extractability of an intersection of spaces	271
18.7. Sequential completion of extractable spaces	271
Part 4. Differential Calculus	273
Chapter 19. Differentiable Mappings	275
19.1. Differentiable mappings	275
19.2. Differentiability, continuity and linearity	277
19.3. Differentiation and change of topology or restriction	279
19.4. Mean value theorem	281
19.5. Bounds on a real differentiable mapping	284
19.6. Differentiation of a composite mapping	286
19.7. Differential of an inverse mapping	289
19.8. Inverse mapping theorem	290
Chapter 20. Differentiation of Multivariable Mappings	295
20.1. Partial differentiation	295
20.2. Differentiation of a multilinear or multi-component mapping	298
20.3. Differentiation of a composite multilinear mapping	300
Chapter 21. Successive Differentiations	303
21.1. Successive differentiations	303
21.2. Schwarz's symmetry principle	305
21.3. Successive differentiations of a composite mapping	308
Chapter 22. Derivation of Functions of One Real Variable	313
22.1. Derivative of a function of one real variable	313
22.2. Derivative of a real function of one real variable	315
22.3. Leibniz formula	319
22.4. Derivatives of the power, logarithm and exponential functions	320
Bibliography	325
Cited Authors	331
Index	335