Chapter 1

Introduction

"Meshless" methods are alternative techniques to the finite element method in solving partial differential equations. While the finite element method derives an approximation based on the elements, using shape functions, the meshless methods allow us to derive an approximation at any point; thanks to the information provided by the surrounding nodes. In these approaches the concept of element is thus not used any more. Connectivity between the nodes is not defined any more by the mesh but only by the concepts of "vicinity" or "field of influence." These methods were developed with the aim of avoiding the numerical problems involved in mesh construction. These problems have been discussed in many studies; it is, for example, a question of simulation of manufacturing processes such as extrusion, injection, or setting forms by removal of matter where it is necessary to face extremely large distortions of the mesh. In other processes such as foundry, drilling, or laser welding, precisely knowing the position of the interface between the solid phase and the liquid phase is essential. In the simulation of processes such as cutting by adiabatic shearing which involves a localized deformation, possibly accompanied by the propagation of a fissure, it is necessary to carry out the simulation without the mesh being conceived influencing the direction of propagation of the shear band or the fissure. The appearance of a localized deformation requires a finer representation of the solution in certain areas of the domains, and it is thus necessary to be able to refine the mesh easily without the geometrical constraints known within the framework of finite elements (mainly in 3D) and the problems related to precise projection of the fields between the two meshes. The objective of the meshless methods is to eliminate the structure of the mesh and to build the approximation starting only from the nodes. Although structures with a geometrical character are necessary (to build node connectivity for the integration of the weak form associated with the equation to be solved and so on), these do not interfere, in general, with the quality of the solution and thus can be built independently. Even after being proposed at the end

of 1970s, the "meshless" methods had to wait approximately 15 years before having a real development and an interest within the scientific community.

In the interval, little passion had been shown for them because of the numerous difficulties presented by the first techniques. The first "meshless" method seems to be the so-called smooth particle hydrodynamics (SPH) method (Lucy 1977), which was initially used to model astronomical phenomena in unbounded domains. This method, based on an approximation using the properties of the convolution product, has two disadvantages: low consistency and difficulty associated with the imposition of boundary conditions. In 1992, Nayrolles, Touzot and Villon proposed using a local approximation of least squares in a new method called the "diffuse elements method" (DEM). In 1994, Belytschko *et al.* proposed the "element-free Galerkin" (EFG) method based on the same principles as the preceding one but using "exact" derivatives of the shape functions. The method known as the "reproducing kernel particle method" (RKPM) introduced by Liu *et al.* in 1995 is an extension of the SPH method but with the reproduction of linear fields or polynomials of higher order being introduced, thanks to the correction function affecting the kernel function used in SPH method. Finally, the so-called partition of unity method introduced by Babuska in about 1996 is a general principle allowing us to enrich any function associated with a problem involving known physics, within the framework of finite elements and of meshless methods, by adding additional unknowns in the global system of equations. Thus, particular functions such as discontinuous functions and singular functions can be reproduced.

Lastly, more recently, the natural element method (NEM) rests on principles completely different from the previous ones. This method is halfway between meshless methods and the finite element method. The NEM proposes an interpolation based on the concepts of the Voronoi diagram and its natural neighbors. The Voronoi diagram associated with a cloud of nodes distributed over the domain to be studied is the Delaunay dual mesh. Thus, a mesh is being used for the construction of the interpolation. However, as the examples presented in this chapter show, the quality of interpolation produced does not depend on the form of the triangles (2D problems) or tetrahedrons (3D problems) present in the Delaunay mesh. The latter is built in a systematic way without requiring repositioning of nodes. With NEM the choice of support of shape functions is automatic and optimal in the sense that node vicinity is taken into account as much as possible to define the interpolation. With regard to the imposition of boundary conditions, for *convex domains*, it is direct and proceeds as the finite elements: the influence of internal nodes on a given domain is cancelled on the edges of the latter. The NEM cumulates the advantages of meshless methods and finite element approaches even if, with respect to the latter, a surcharge exists for the construction of the interpolation.

To extend these characteristics to the *non-convex domains* two strategies exist. The first approach based on the alpha forms makes it possible to introduce a description of

the border in a very flexible way if the latter remains slightly non-convex. In the case of strongly non-convex fields, the constrained NEM (CNEM) proposes to build the interpolation on a constrained Voronoi diagram, which is the constrained Delaunay dual mesh. For the second approach, in addition to the node cloud, a valid description of the border of the field must be introduced. The Delaunay mesh is constrained with respect to this border.

The purpose of this text is to describe the technique of natural elements in its context, i.e. compared to the techniques of finite elements type, which have proved reliable for many years, but also compared to other techniques with and without meshes. Both advantages and disadvantages have been listed. This book has been written with a teaching purpose to be used by both professionals and students at Master's level. Many examples have been discussed to illustrate our remarks in order to show the potentialities of the approach. The majority of these examples will be from the framework of simulation of methods of working where the application of the meshless techniques takes all its direction owing to the great material transformations that seriously compromise the effectiveness of the techniques based on the existence of a mesh in Lagrangian formulation (or updated Lagrangian).

To better understand the context how the NEM appeared, we will revisit the main so-called meshless techniques that preceded it, for which these techniques will be described and discussed briefly and many references will be provided to allow the reader to further develop their comprehension.

1.1. SPH method

The SPH (Smooth Particle Hydrodynamics) method was introduced for solving astrophysics models. It is based on an approximation built starting from an integral of a convolution product

$$\boldsymbol{u}^h(\boldsymbol{x}) = \int \boldsymbol{u}(\boldsymbol{y}) W(\boldsymbol{x}-\boldsymbol{y}, h) d\Omega_{\boldsymbol{y}}, \qquad [1.1]$$

where the kernel function has the following properties:

– $\lim_{h\to 0} \int \boldsymbol{u}(\boldsymbol{y}) W(\boldsymbol{x}-\boldsymbol{y}, h) \mathrm{d}\Omega_{\boldsymbol{y}} = \boldsymbol{u}(\boldsymbol{x})$;

– $\int W(\boldsymbol{x}-\boldsymbol{y}, h) \mathrm{d}\Omega_{\boldsymbol{y}} = 1$;

– W has a compact support;

– W is decreasing with distance;

– $W(\boldsymbol{x}-\boldsymbol{y}, h) \in \mathcal{C}^p(\mathbb{R}^n), \quad p \geq 1.$

The first property expresses that the function core tends toward the Dirac distribution, which is the limit in which equation [1.1] makes sense. The second

property ensures zero-order consistency of the approximation. The third ensures the locality of the approximation and thus after discretization it will lead to a sparse linear system. The last condition allows us to obtain certain regularity in the resulting approximation.

One of the most used kernel functions is that using a Gaussian

$$W(\boldsymbol{x}, h) = \frac{1}{(\pi h^2)^{n/2}} \exp\Big[- \frac{\boldsymbol{x}^2}{h^2} \Big]; \qquad [1.2]$$

even if other kernels are also often used (splines, etc.).

In general, the support of these kernel functions is circular (in 2D) or spherical (in 3D), but it is also possible to make it ellipsoidal or even rectangular (Figure 1.1) with the introduction of tensorial products:

$$W(\boldsymbol{x} - \boldsymbol{x_I}) = W(x - x_I) W(y - y_I). \qquad [1.3]$$

The integral in equation [1.1] can be discretized using a nodal quadrature

$$\boldsymbol{u}^h(\boldsymbol{x}) = \sum_{I : \boldsymbol{x} \in \Omega_I} W(\boldsymbol{x} - \boldsymbol{x_I}) \Delta V_I \boldsymbol{u}_I, \qquad [1.4]$$

with ΔV_I the volume associated with each node. The approximation can thus be rewritten in its more usual form:

$$\boldsymbol{u}^h(\boldsymbol{x}) = \sum_{I : \boldsymbol{x} \in \Omega_I} \Phi_I(\boldsymbol{x}) \boldsymbol{u}_I, \qquad [1.5]$$

where the functions Φ_I represent the shape functions of the approximation.

The fact of having a poor consistency explains why SPH approximation was especially used in the discretization of the continuous models (partial differential equations) in its strong formulation by means of a collocation scheme.

Many difficulties have been listed concerning the use of SPH approximation. These difficulties justified many works trying to circumvent these difficulties with more or less success. We can recount some of them: imposition of the essential boundary conditions (Dirichlet) due to a non-interpolant character of the approximation (equation [1.5]), i.e. the shape functions associated with the interior nodes, the support of which has a non-zero intersection with the edge of the field, are not cancelled

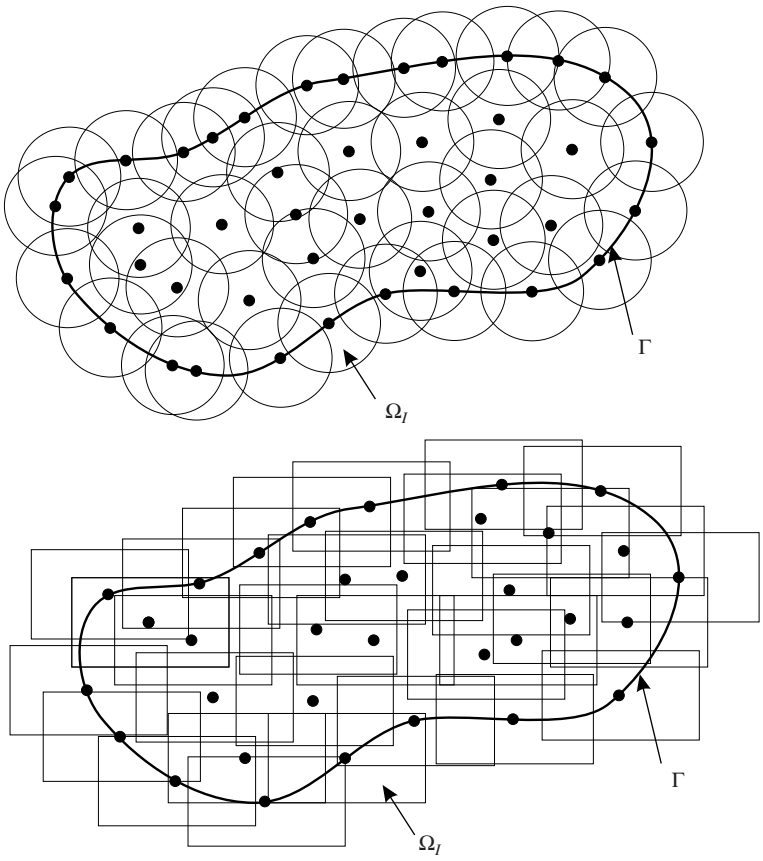

Figure 1.1. *Recovery of a 2D field*

out on the domain boundary; instabilities observed in the solids subjected to states of tensile stress that justified the introduction of the "stresses points" among other solutions; parasitic modes of deformation; and the inconsistency mentioned already.

These problems of inconsistency justified the proposal of corrections of the kernels for ensuring consistency, a step that led to RKPM.

1.2. RKPM method

To simplify the explanation we consider here a unidimensional domain as the support of the problem to be solved (all the results can be extended to the 2D or 3D case). The points in this domain will be represented by x or s.

1.2.1. *Conditions of reproduction*

The approximation $u^h(x)$ of $u(x)$ is derived from the convolution product:

$$u^h(x) = \int_\Omega w(x-s,h)u(s)\mathrm{d}\Omega, \qquad [1.6]$$

where $w(x-s,h)$ is the kernel function and h a parameter that controls the support of the approximation.

The main idea in the RKPM is to force the approximation to reproduce an unspecified function. By simplicity we will suppose that the function, which we want to reproduce exactly, is written as the sum of a polynomial part and non-polynomial part $u^e(x)$:

$$u^h(x) = a_0 + a_1 x + \cdots + a_n x^n + a_{n+1}u^e(x). \qquad [1.7]$$

In the following section, we will be discussing regarding the properties that the kernel function will have to satisfy in order to define an approximation which will be able to reproduce exactly the function in equation [1.7].

From equation [1.6], the reproduction of a constant function a_0 is written as

$$\int_\Omega w(x-s,h)a_0\mathrm{d}\Omega = a_0, \qquad [1.8]$$

which implies

$$\int_\Omega w(x-s,h)\mathrm{d}\Omega = 1, \qquad [1.9]$$

which is none other than the partition of unity.

The condition to be verified in order to reproduce a linear function $u^a(x) = a_0 + a_1 x$ is expressed in the same way by

$$\int_\Omega w(x-s,h)(a_0 + a_1 s)\mathrm{d}\Omega = a_0 + a_1 x. \qquad [1.10]$$

Using the partition of unity (equation [1.9]), equation [1.10] can be modified to the form

$$\begin{cases} \int_\Omega w(x-s,h)\mathrm{d}\Omega = 1 \\ \int_\Omega w(x-s,h)s\mathrm{d}\Omega = x \end{cases} \qquad [1.11]$$

implying the linear consistency of the approximation. By repeating procedure, we can derive the n-order reproduction

$$\begin{cases} \int_\Omega w(x-s,h)\mathrm{d}\Omega = 1 \\ \int_\Omega w(x-s,h)s\mathrm{d}\Omega = x \\ \qquad\qquad \vdots \\ \int_\Omega w(x-s,h)s^n\mathrm{d}\Omega = x^n. \end{cases} \quad [1.12]$$

Consequently, by rewriting the function present in equation [1.7] we get

$$\int_\Omega w(x-s,h)(a_0 + a_1 s + \cdots + a_n s^n + a_{n+1}u^e(s))\mathrm{d}\Omega \;=$$

$$a_0 + a_1 x + \cdots + a_n x^n + a_{n+1}u^e(x), \quad [1.13]$$

from which we deduce

$$\begin{cases} \int_\Omega w(x-s,h)\mathrm{d}\Omega = 1 \\ \int_\Omega w(x-s,h)s\mathrm{d}\Omega = x \\ \qquad\qquad \vdots \\ \int_\Omega w(x-s,h)s^n\mathrm{d}\Omega = x^n \\ \int_\Omega w(x-s,h)u^e(s)\mathrm{d}\Omega = u^e(x). \end{cases} \quad [1.14]$$

In the original procedure suggested by Liu *et al.* [LIU 95], only polynomial consistency of degree n was imposed. However, this procedure cannot be directly generalized to impose the reproduction of a generic non-polynomial function $u^e(x)$.

1.2.2. *Correction of the kernel*

We will represent by $u^r(x)$ the function of approximation verifying the conditions stated in the system of equations [1.14]. Normally, the kernel function is taken in the form of a cubic spline function, and consequently, the equations [1.14] are not satisfied. Liu *et al.* [LIU 95] proposed the introduction of a correction function $C(x, x-s)$ to satisfy all the conditions of reproduction. In our case, where we want to reproduce any polynomial or non-polynomial function also, we will consider the more general form $C(x, s, x-s)$, the relevance of which will be discussed later. Thus, $u^r(x)$ can be expressed by [TRU 05]:

$$u^r(x) = \int_\Omega C(x,s,x-s)w(x-s,h)u(s)\mathrm{d}\Omega, \quad [1.15]$$

where $C(x, s, x-s)$ is sought in the form

$$C(x, s, x-s) = \mathbf{H}^T(x, s, x-s)\mathbf{b}(x), \tag{1.16}$$

where $\mathbf{H}^T(x, s, x-s)$ represents the vector containing the functions considered in the approximation basis and $\mathbf{b}(x)$ is another vector whose components are unknown functions that will be determined to verify the conditions of reproduction. Thus, equation [1.14] can be rewritten as:

$$\begin{cases} \int_\Omega \mathbf{H}^T(x, s, x-s)\mathbf{b}(x)w(x-s, h)\mathrm{d}\Omega = 1 \\ \int_\Omega \mathbf{H}^T(x, s, x-s)\mathbf{b}(x)w(x-s, h)s\mathrm{d}\Omega = x \\ \qquad\qquad \vdots \\ \int_\Omega \mathbf{H}^T(x, s, x-s)\mathbf{b}(x)w(x-s, h)s^n\mathrm{d}\Omega = x^n \\ \int_\Omega \mathbf{H}^T(x, s, x-s)\mathbf{b}(x)w(x-s, h)u^e(s)\mathrm{d}\Omega = u^e(x). \end{cases} \tag{1.17}$$

In fact, the conditions of reproduction must be imposed in discrete form. To do so, we will consider N points (also called nodes) allowing us to calculate the discrete form of equation [1.17]:

$$\begin{cases} \sum_{i=1}^N \mathbf{H}^T(x, x_i, x-x_i)\mathbf{b}(x)w(x-x_i, h)\Delta x_i = 1 \\ \sum_{i=1}^N \mathbf{H}^T(x, x_i, x-x_i)\mathbf{b}(x)w(x-x_i, h)x_i\Delta x_i = x \\ \qquad\qquad \vdots \\ \sum_{i=1}^N \mathbf{H}^T(x, x_i, x-x_i)\mathbf{b}(x)w(x-x_i, h)x_i^n\Delta x_i = x^n \\ \sum_{i=1}^N \mathbf{H}^T(x, x_i, x-x_i)\mathbf{b}(x)w(x-x_i, h)u^e(x_i)\Delta x_i = u^e(x), \end{cases} \tag{1.18}$$

whose matrix form is

$$\left[\sum_{i=1}^N \mathbf{R}(x_i)\mathbf{H}^T(x, x_i, x-x_i)w(x-x_i, h)\Delta x_i\right]\mathbf{b}(x) = \mathbf{R}(x), \tag{1.19}$$

where $\mathbf{R}(x)$ represents the vector of reproduction,

$$\mathbf{R}^T(x) = \left[1, x, \ldots, x^n, u^e(x)\right]. \tag{1.20}$$

Equation [1.19] enables us to calculate the vector $\mathbf{b}(x)$,

$$\mathbf{b}(x) = \mathbf{M}(x)^{-1}\mathbf{R}(x), \tag{1.21}$$

where the so-called moment matrix $\mathbf{M}(x)$ is defined by:

$$\mathbf{M}(x) = \sum_{i=1}^{N} \mathbf{R}(x_i)\mathbf{H}^T(x, x_i, x - x_i)w(x - x_i, h)\Delta x_i. \qquad [1.22]$$

This matrix differs slightly from that obtained in [LIU 95].

1.2.3. *Discrete form of the approximation*

The discrete form $u^r(x)$ of $u^h(x)$ is obtained from equations [1.15], [1.16] and [1.21]:

$$\begin{aligned} u^r(x) &\cong \sum_{i=1}^{N} \mathbf{H}^T(x, x_i, x - x_i)\mathbf{M}(x)^{-1}\mathbf{R}(x)w(x - x_i, h)u(x_i)\Delta x_i \\ &= \sum_{i=1}^{N} \psi_i(x)u_i, \end{aligned} \qquad [1.23]$$

where ψ_i is the shape function associated with the enriched RKPM approximation:

$$\psi_i(x) = \mathbf{H}^T(x, x_i, x - x_i)\mathbf{M}(x)^{-1}\mathbf{R}(x)w(x - x_i, h)\Delta x_i. \qquad [1.24]$$

As in the most standard version of the RKPM, we take $\Delta x_i = 1$. Although various quadratures exist, the choice of the quadrature does not affect the precision of the constructed approximation.

If this method allows us to overcome a certain number of difficulties present in the SPH method, the difficulty related to the imposition of the essential boundary conditions remains untouched. The gain on the side of consistency allowed the use of RKPM approximations within the framework of discretizations of the weak (often variational) formulations of partial differential equations. Although this possibility of working on weak formulations seems to be a positive point, it hides a new difficulty associated with the integration of the weak forms. Integration requires a decomposition of the domain and use of a suitable quadrature formula. Moreover, in certain cases, the non-polynomial character of the resulting shape functions makes this integration delicate. This subject has also motivated many studies.

It is possible to prove that the RKPM, which we have just described briefly, is completely equivalent to another family of methods. These methods will be described later and are based on the use of approximations making use of the moving least squares (*MLS*).

1.3. MLS based approximations

We now will consider the approximation:

$$\mathbf{u}^h(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}), \quad [1.25]$$

with $\mathbf{p}^T(\mathbf{x})$ a polynomial base. For example, $\mathbf{p}^T(\mathbf{x}) = [1, x, y, xy]$ and $\mathbf{p}^T(\mathbf{x}) = [1, x, y, xy, x^2, y^2]$, respectively, represent a linear and quadratic base in the 2D case and $\mathbf{a}(\mathbf{x})$ represents a vector with unknown coefficients. To determine $\mathbf{a}(\mathbf{x})$, we will define the functional calculus J which will have to be minimized with respect to $\mathbf{a}(\mathbf{x})$ [NAY 92]:

$$J = \frac{1}{2}\sum_{i=1}^{n} w_i(\mathbf{x}) \left[\mathbf{p}^T(\mathbf{x}_i)\mathbf{a}(\mathbf{x}) - u_i\right]^2, \quad [1.26]$$

where u_i are the nodal unknown associated with nodes $\mathbf{x}_i$ neighboring points of $\mathbf{x}$ and $w_i(\mathbf{x})$ is a weight function whose value decreases with the distance between $\mathbf{x}_i$ and $\mathbf{x}$ (refer to [BEL 98a] to understand the main properties of this function as well as the most used weight functions). The minimization of J with respect to the coefficients $a_j(\mathbf{x})$ led to:

$$\frac{\partial J}{\partial a_j(\mathbf{x})} = \sum_{k=1}^{n} a_k \left[\sum_{i=1}^{n} w_i(\mathbf{x}) p_j(\mathbf{x}_i) p_k(\mathbf{x}_i)\right] - \sum_{i=1}^{n} w_i(\mathbf{x}) p_j(\mathbf{x}_i) u_i = 0, \quad [1.27]$$

which led to the linear system:

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\mathbf{u}, \quad [1.28]$$

where the matrices $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are defined by:

$$A_{jk}(\mathbf{x}) = \sum_{i=1}^{n} w_i(\mathbf{x}) p_j(\mathbf{x}_i) p_k(\mathbf{x}_i), \quad [1.29]$$

$$B_{ij}(\mathbf{x}) = w_i(\mathbf{x}) p_j(\mathbf{x}_i). \quad [1.30]$$

While replacing $\mathbf{a}(\mathbf{x})$ in equation [1.25], we obtain:

$$u^h(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{u}, \quad [1.31]$$

from which we can identify the shape functions of the approximation:

$$\boldsymbol{\psi}^T(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}). \quad [1.32]$$

The only difference between the diffuse approximation and approximation used in the technique known as *EFG* resides in the evaluation of derivatives of the shape functions. In the first technique, only the functions of the approximation base contained in vector $\mathbf{p}^T(\mathbf{x})$ in equation [1.32] are derived, while in the second all the terms depending on $\mathbf{x}$ are there. It amounts to the saying that not only are the derivatives of the functions contained in the base of approximation considered but also the derivatives of the coefficients $a_j(\mathbf{x})$ present in the approximation.

1.4. Final note

Although we have summarized only the most known methods here, there exist various techniques which can be listed as pertaining to the family of meshless techniques. Since many works and articles in specialized papers have been devoted to these techniques, we do not want to expand on them further. We will quote simply some other techniques: generalized finite differences, the "h-p clouds;" finite sphere methods; and the methods based on the partition of unity such as the generalized finite elements or even the extended finite elements (X-FEM) methods which, although based on the finite element method, succeed in freeing themselves from certain difficulties related to the management of interfaces or discontinuities, fixed or moving without calling on the traditional remeshing techniques.